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Density Matrices and the Weak Quantum Numbers

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Abstract Pure density matrices are idempotent, Hermitian, and have trace 1. The idempotency equation for an $N \times N$ matrix gives N^2 quadratic equations in N^2 unknowns. We consider a subalgebra of the 6×6 complex matrices defined by the permutation group on three elements. We show that the Hermitian, idempotent members of this subalgebra have quantum numbers that exactly match the weak hypercharge and weak isospin quantum numbers of the left and right handed elementary fermions.

The use of the permutation group on 3 elements suggests that the elementary fermions are composite, or satisfy the Tripled Pauli statistics found by Lubos Motl, instead of the expected Fermi-Dirac statistics.

Keywords density matrix · weak hypercharge · weak isospin · Tripled Pauli statistics · preon

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1 Introduction

Pure density matrices ρ satisfy the idempotency equation $\rho^2 = \rho$. An idempotency relation defines a set of coupled quadratic equations. For example, the spin-1/2 density matrix corresponding to spin in the $\hat{u}$ direction is given by $\rho_u = (1 + \hat{u} \cdot \boldsymbol{\sigma})/2$ where $\boldsymbol{\sigma}$ is the vector of Pauli spin operators. Written out using the Pauli spin matrices, this gives:

$$\rho_u = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{1}$$

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From the idempotency equation $\rho_u^2 = \rho_u$, we get a set of 4 coupled quadratic equations in the 4 unknowns a_{jk} as follows:

$$\begin{aligned} a_{11} &= a_{11}a_{11} + a_{12}a_{21} \\ a_{12} &= a_{11}a_{12} + a_{12}a_{22} \\ a_{21} &= a_{21}a_{11} + a_{22}a_{21} \\ a_{22} &= a_{21}a_{12} + a_{22}a_{22}. \end{aligned} \tag{2}$$

Four linear equations in four unknowns would have either no solution, a single unique solution, or an infinite number of solutions. For the case of coupled quadratic equations, more complicated solution sets are possible. Solving this set of equations gives us hints in how to obtain physically relevant solutions from the general solutions of an idempotency relation.

Two easy solutions to Eq. (2) are the 0 and 1 matrices:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3}$$

These matrices are not pure density matrices because their traces are not equal to 1. However, we can interpret them as the density matrices representing the state with no particles and the state with two particles of opposite spin. Then the trace can be thought of as giving a count of the number of particles represented.

The remaining solutions to Eq. (2) have trace 1. Pure density matrices must be Hermitian but these remaining solutions also include non Hermitian matrices. We can parameterize all the trace 1 solutions (both Hermitian and non Hermitian), by choosing two unit vectors $\hat{u}$ and $\hat{v}$ with $\hat{u} + \hat{v} \neq 0$. Then any of the remaining solutions (i.e. idempotent 2×2 matrices with trace 1) can be written uniquely as

$$2\rho_u\ \rho_v/(1 + \hat{u}\cdot\hat{v}) = (1 + \hat{u}\cdot\boldsymbol{\sigma})(1 + \hat{v}\cdot\boldsymbol{\sigma})/(2 + 2\hat{u}\cdot\hat{v}). \tag{4}$$

This is a real multiple of the product of two (Hermitian) pure density matrices. The Hermitian solutions have $\hat{u} = \hat{v}$.

The first generation left and right handed elementary fermions, and their antiparticles, have the following weak hypercharge (t_0) and weak isospin (t_3) quantum numbers, particles on the left and anti-particles on the right:

	t_0	t_3		t_0	t_3
ν_L	-1	$+1/2$	$\bar{\nu}_R$	$+1$	$-1/2$
ν_R	0	0	$\bar{\nu}_L$	0	0
d_L	$+1/3$	$-1/2$	$\bar{d}_R$	$-1/3$	$+1/2$
d_R	$-2/3$	0	$\bar{d}_L$	$+2/3$	0
e_L	-1	$-1/2$	$\bar{e}_R$	$+1$	$+1/2$
e_R	-2	0	$\bar{e}_L$	$+2$	0
u_L	$+1/3$	$+1/2$	$\bar{u}_R$	$-1/3$	$-1/2$
u_R	$+4/3$	0	$\bar{u}_L$	$-4/3$	0

(5)

The quantum numbers of the other generations are the same. Note that each pair of quantum numbers appears twice, once for a particle and once again, negated, for an anti-particle.

Our equations will provide the quantum numbers for the fermions with weak hypercharge non negative; some of these are particles while others are anti-particles. In general, if one obtains the quantum numbers of the particles with positive weak hypercharge by solving the idempotency equation $\rho^2 = \rho$, then a set of equations that will give the negative weak hypercharge quantum numbers is simply $\rho^2 = -\rho$.

The requirement that a pure density matrix satisfy the equation $\rho^2 = +\rho$ arises from the assumption that the elementary particles can be represented by a state vector, and that the state vector can always be normalized. While state vectors are fully adequate for the usual formulation of quantum mechanics, they are somewhat clunky when modeling non Hermitian states. In a following paper we will use these concepts to model the generation structure of the elementary fermions.

While our interest is in the permutation group on 3 elements, we will first discuss the considerably simpler case of the permutation group on 2 elements.

2 Permutations of 2 elements

We will write the group with two elements as $\{R, G\}$, and the permutation group on them as $\{I, S\}$. The action of the permutation group on the two elements is defined as follows:

$$\begin{array}{c|c} & R\ G \\ \hline I & R\ G \\ S & G\ R \end{array} \tag{6}$$

so I is the identity and S swaps the two elements. The identity I is the trivial symmetry and can be thought of as representing the $U(1)$ symmetry. The Pauli spin matrix σ_z squares to 1. This is analogous to the fact that $S^2 = I$, so we will think of S as representing weak isospin $SU(2)$.

We can write the identity I in two ways as a product of two group elements: $I = II$, and $I = SS$. To convert this into a quadratic equation, we write $I = I^2 + S^2$ with I and S now thought of as complex numbers. In terms of quantum mechanics, we can think of this as writing the transition amplitude I as the sum of two processes, the first with the particle undertaking two transitions each with amplitudes I, the second with the particle undertaking two transitions with amplitude S.

Similarly, we can write S in two ways as $S = IS$, and $S = SI$. This gives the quadratic equation $S = 2IS$. Our complete set of coupled quadratic equations is:

$$\begin{aligned} I &= I^2 + S^2, \\ S &= 2IS. \end{aligned} \tag{7}$$

There are four solutions to these two coupled quadratic equations. They correspond to the quantum numbers of the leptons as follows:

$$\begin{array}{c|cc} & 2I & S \\ & t_0 & t_3 \\ \hline \bar{\nu}_R & +1 & -1/2 \\ \bar{\nu}_L & 0 & 0 \\ \bar{e}_R & +1 & +1/2 \\ \bar{e}_L & +2 & 0 \end{array}. \tag{8}$$

If we were to replace the usual weak hypercharge t_0 with its contribution to electric charge, $t_0/2$, the correspondence is exact; the solutions to the coupled quadratic equations defined by the permutation group on two elements gives the $t_0/2$ and t_3 quantum numbers of the leptons.

3 Permutations of 3 elements

For the permutation group on 3 elements, we will use the elements $\{R, G, B\}$ and the group elements $\{I, J, K, R, G, B\}$ with their action defined as:

$$\begin{array}{c|ccc} & R & G & B \\ \hline I & R & G & B \\ J & G & B & R \\ K & B & R & G \\ R & R & B & G \\ G & B & G & R \\ B & G & R & B \end{array} \tag{9}$$

The permutation group is then

$$\begin{array}{c|cccccc} & I & J & K & R & G & B \\ \hline I & I & J & K & R & G & B \\ J & J & K & I & B & R & G \\ K & K & I & J & G & B & R \\ R & R & G & B & I & J & K \\ G & G & B & R & K & I & J \\ B & B & R & G & J & K & I \end{array} \tag{10}$$

Converting this permutation group into quadratic equations as in the previous section, we obtain:

$$\begin{aligned} I &= I^2 + JK + KJ + R^2 + G^2 + B^2, \\ J &= IJ + JI + K^2 + RG + GB + BR, \\ K &= IK + J^2 + KI + RB + GR + BG, \\ R &= IR + JG + KB + RI + GK + BJ, \\ G &= IG + JB + KR + RJ + GI + BK, \\ B &= IB + JR + KG + RK + GJ + BI. \end{aligned} \tag{11}$$

As with the idempotency problem for 2×2 matrices, there will be discrete solutions as well as continuous solutions.

For the case of the permutation group on 2 elements, we used $I = t_0/2$ and we will retain this assignment. The three elements R, G, and B are all equivalent to the S of the permutation group on 2 elements. We will assign $R + G + B = t_3$.

As a first step in solving these equations, we rewrite them as an equivalent set of six equations:

$$\begin{aligned} I &= I^2 + 2JK + (R^2 + G^2 + B^2), \\ (R+G+B)^2 &= (I+J+K)(1-(I-J-K)), \\ 0 &= (J-K)(1+J+K-2I), \\ (1-3I+(I+J+K))R &= (R+G+B)(J+K), \\ (1-3I+(I+J+K))G &= (R+G+B)(J+K), \\ (1-3I+(I+J+K))B &= (R+G+B)(J+K). \end{aligned} \tag{12}$$

Choosing $I = 1/2$ and $J = -K$ solves the last four of these equations. The two remaining equations reduce to:

$$\begin{aligned} 1/2 &= \pm(R+G+B), \\ J^2 &= -1/8 + (R^2+G^2+B^2)/2. \end{aligned} \tag{13}$$

Since we have solved four equations with only two assignments, the solution space will be at least a 2-manifold. We will parameterize the solutions with complex numbers α, and β. Eventually we find that we can write four 2-manifolds of solutions:

$$\begin{array}{cccccc} I & J & K & R & G & B \\ \hline 1/2 & +\gamma & -\gamma & +1/6+\alpha & +1/6+\beta & +1/6-\alpha-\beta \\ 1/2 & -\gamma & +\gamma & +1/6+\alpha & +1/6+\beta & +1/6-\alpha-\beta \\ 1/2 & +\gamma & -\gamma & -1/6+\alpha & -1/6+\beta & -1/6-\alpha-\beta \\ 1/2 & -\gamma & +\gamma & -1/6+\alpha & -1/6+\beta & -1/6-\alpha-\beta \end{array}, \tag{14}$$

where $\gamma = \sqrt{\alpha^2 + \beta^2 + \alpha\beta - 1/12}$. The above solutions share $t_0 = 2I = +1$, and have $t_3 = R + G + B = \pm 1/2$. These are the weak quantum numbers of the $\bar{\nu}_R$ and $\bar{e}_R$.

Eliminating the case "$I = 1/2$ and $J = -K$", there are 10 discrete solutions. Six of these show up as two triplets:

$$\begin{array}{cccccc} I & J & K & R & G & B \\ \hline 1/3 & w^{+n}/3 & w^{-n}/3 & 0 & 0 & 0 \\ 2/3 & -w^{+n}/3 & -w^{-n}/3 & 0 & 0 & 0 \end{array} \tag{15}$$

where $w = \exp(2i\pi/3)$ and $n = 0, 1, 2$. All six of these solutions have weak isospin zero. The two triplets differ in weak hypercharge with $t_0 = 2I = +2/3$ and $t_0 = 2I = +4/3$. These are the weak quantum numbers of the d_R and u_R.

The remaining four discrete solutions have different combinations of weak hypercharge and weak isospin:

$$\begin{array}{cccccc} I & J & K & R & G & B \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1/6 & 1/6 & 1/6 & -1/6 & -1/6 & -1/6 \\ 1/6 & 1/6 & 1/6 & +1/6 & +1/6 & +1/6 \end{array} \tag{16}$$

The first two of these solutions have weak isospin 0 and weak hypercharge 0 and 2. These are the quantum numbers of the ν_R (or $\bar{\nu}_L$) and the $\bar{e}_L$. The last two solutions share weak hypercharge 1/3 and have weak isospin $\pm 1/2$; these are the quantum numbers of the d_L and u_L.

The complete set of solutions to the 6 coupled quadratic equations, and their assignment to the first generation fermions are as follows:

	I	J	K	R	G	B
$\bar{\nu}_L/\nu_R$	0	0	0	0	0	0
d_L	1/6	1/6	1/6	$-1/6$	$-1/6$	$-1/6$
u_L	1/6	1/6	1/6	$+1/6$	$+1/6$	$+1/6$
$\bar{d}_L$	1/3	$w^{+n}/3$	$w^{-n}/3$	0	0	0
$\bar{\nu}_R$	1/2	$\pm\gamma$	$\mp\gamma$	$-1/6+\alpha$	$-1/6+\beta$	$-1/6-\alpha-\beta$
$\bar{e}_R$	1/2	$\pm\gamma$	$\mp\gamma$	$+1/6+\alpha$	$+1/6+\beta$	$+1/6-\alpha-\beta$
u_R	2/3	$-w^{+n}/3$	$-w^{-n}/3$	0	0	0
$\bar{e}_L$	1	0	0	0	0	0

(17)

where w and γ are as above.

This result is lacking in that we have three different choices each for $\bar{d}_L$ and u_R, and an infinite number for $\bar{\nu}_R$ and $\bar{e}_R$. We can think of these extra solutions as being in analogy with the non Hermitian solutions to the 2×2 matrix idempotency problem Eq. (2). To get rid of them, we need to define "Hermiticity" for the solutions.

4 Hermiticity

Since the six quadratic equations are generated from the permtuation group on three elements, we begin with matrices that represent that group. For the even permutations, we have:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \; K = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{18}$$

No two of these three matrices is non zero in the same position. Consequently, we can multiply each by its corresponding complex number I, J, and K and then assemble the resulting three matrices them into a single complex matrix P_0:

$$P_0 = \begin{pmatrix} I & J & K \\ K & I & J \\ J & K & I \end{pmatrix}. \tag{19}$$

This is a 1-circulant matrix. That is, each row is the same as the previous row rotated one position to the right. Similarly, we can assemble the odd permutations into a matrix P_1:

$$P_1 = \begin{pmatrix} R & B & G \\ B & G & R \\ G & R & B \end{pmatrix}, \tag{20}$$

a 2-circulant matrix.

The 1-circulant 3×3 matrices form a subalgebra of the 3×3 matrices; the product or sum of any two such matrices is a matrix of the same sort. Products of two 2-circulant matrices are a 1-circulant, and the product of a 1-circulant and a 2-circulant is 2-circulant. These are the same rules that apply to the diagonal (1-circulant) and off-diagonal (2-circulant) elements of a 2×2 matrix. Consequently, we can assemble P_0 and P_1 into a 6×6 matrix:

$$P = \left(\begin{array}{ccc|ccc} I & J & K & R & B & G \\ K & I & J & B & G & R \\ J & K & I & G & R & B \\ \hline R & B & G & I & J & K \\ B & G & R & K & I & J \\ G & R & B & J & K & I \end{array}\right) \tag{21}$$

The six coupled equations Eq. (11) are defined by $P^2 = P$.

Matrices of this form are a subalgebra of the 6×6 complex matrices. That is, they include 0 and 1, and are closed under negation, addition and multiplication. They can be thought of as defining a form of multiplication that operates between two 6-element complex vectors.

These 6×6 matrices give a natural definition of "Hermiticity" to the solutions to the 6 coupled quadratic equations of Eq. (11). We will say that a solution is Hermitian if the related 6×6 matrix is Hermitian. Examining Eq. (21), we find Hermiticity requires that I, R, G, and B must be real and that J and K must be complex conjugates of each other.

Applying this definition of Hermiticity to the 8 classes of solutions given in Eq. (17), we find that each elementary fermion is represented by a unique Hermitian solution. The assignments of the elementary particles,

	I	J	K	R	G	B
$\bar{\nu}_L/\nu_R$	0	0	0	0	0	0
d_L	1/6	1/6	1/6	−1/6	−1/6	−1/6
u_L	1/6	1/6	1/6	+1/6	+1/6	+1/6
$\bar{d}_L$	1/3	1/3	1/3	0	0	0
$\bar{\nu}_R$	1/2	$\pm i\sqrt{3}/6$	$\mp i\sqrt{3}/6$	−1/6	−1/6	−1/6
$\bar{e}_R$	1/2	$\pm i\sqrt{3}/6$	$\mp i\sqrt{3}/6$	+1/6	+1/6	+1/6
u_R	2/3	−1/3	−1/3	0	0	0
$\bar{e}_L$	1	0	0	0	0	0

(22)

are unique up to the choice of the sign of the imaginary unit.

Thus the requirement of Hermiticity has decreased the solution set to the idempotency problem down to just the quantum numbers of the left and right handed elementary fermions. This is in analogy to how the pure density matrices of the Pauli algebra are defined in terms of the idempotent 2×2 matrices. The only difference is that the pure density matrices of the Pauli algebra also satisfy the restriction that their trace is 1.

This lack of a requirement on the trace suggests that if we are to think of the left and right handed elementary fermions as being defined by (generalized) pure density matrices we cannot assume that they are the density matrices of single quantum states. Instead, they would have to be composite. The traces of the Hermitian solutions are given by $6I$ and are integers from 0 to 6. With the interpretation of the trace as giving the number of preons in a generalized pure density matrix minus the number of anti-preons, we expect that the left and right handed elementary fermions will be composite with 6 preons / anti-preons.

5 The Fermion Cube

Plotting the weak hypercharge and weak isospin quantum numbers of the elementary fermions, we find that they appear as a tilted cube. The leptons are found on the corners of the cube while the quarks appear along four edges of the cube. See Fig. (1), the "Fermion cube."

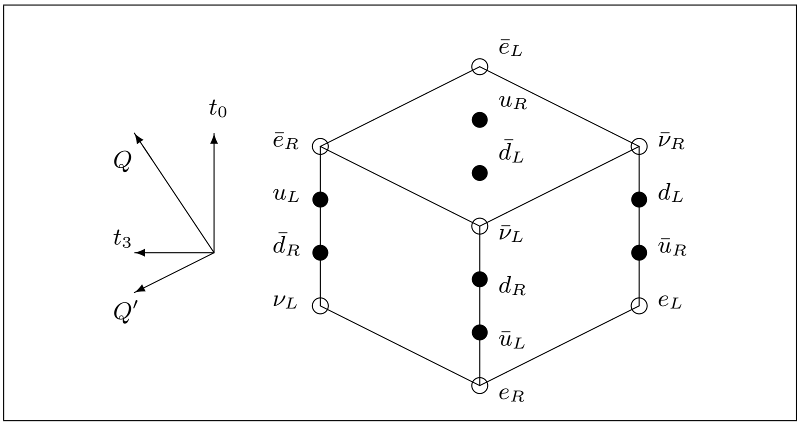

Fig. 1 Weak hypercharge, t_0, and weak isospin, t_3, quantum numbers plotted for the first generation standard model quantum states. Leptons are hollow circles and quarks ($\times 3$) are filled circles. Electric charge, Q, and neutral charge, Q' also shown.

Unlike the leptons, quarks carry color charge. This is not graphed in Fig. (1) and so each quark (filled) dot has multiplicity 3. This pattern, two lepton singletons (one a neutrino, the other a charged lepton) surrounding two quark triplets (one up, the other down), is suggestive of a substructure where the quarks and leptons are composed of 3 preons each. The idea is that the leptons are composed of three, more or less identical, preons with the preon type distinguishing between the leptons. The quarks are composed of mixtures.

This author proposed such a preon model in an unpublished paper[1] that used the idempotent structure of Clifford algebras to count the number of hidden dimensions required to produce the left and right handed elementary fermions. At that time the author believed this preon model was an original idea, but the idea is called a "resurrected preon model"[2] by Gerald Rosen in an article referencing the author's rewrite[3, 4] of Yoshio Koide's charged lepton mass equation[5, 6].

Presumably the reason this preon model required resurrection is that it is difficult to see how to assemble three identical preons into a fermion with spin-1/2. One possible solutions is a modification of the statistics obeyed by quantum particles.

6 Tripled Pauli Statistics

It has long been known (Price's theorem[7]) that black holes exponentially approach a condition where the only numbers that characterize them are their mass, spin, and their electric charge. According to the laws of quantum mechanics, this exponential decay will be accomplished through Hawking radiation[8], the emission of quantum objects. One can make a slight leap of faith and suppose that knowing more about this classical exponential decay will give us information about the quantum particles that are emitted.

This observation motivates[9] an examination of the quasinormal modes of vibration of black holes, that is, the modes that can be characterized as following an $\exp(-at)\sin(\omega t)$ law. If the decay is due to the emission of only a single type of quantum particle, the sine function defines a frequency ω, that is natural to associate with the energy of the emitted particles. The exponential function gives the rate at which these particles are emitted. The spectrum of ω is a fundamental characteristic of a black hole.

The correspondence between the classical frequency ω, and the quantum particles making up the Hawking radiation is an example of Bohr's correspondence principle; classical mechanics should be obtained for very large quantum numbers. In this case, quantum transition frequencies should be equal to classical oscillation frequencies. Therefore the frequencies of the quasinormal modes can be thought of as transition frequencies.

To obtain the quasinormal modes of vibration, one solves an equation that is something like Schroedinger's wave equation. The energy will be complex with the imaginary part giving the exponential decay. Of interest are the modes where this decay is as rapid as possible, the "asymptotic" modes.

A consequence of Hawking radiation is that black holes have a temperature. Hawking showed that this temperature is proportional to the area. With quantum mechanics, classical concepts of thermodynamics like temperature are a consequence of statistical mechanics. And in statistical mechanics, one makes the assumption that quantum particles have either Fermi-Dirac or Bose-Einstein statistics. These are slightly different from the Maxwell statistics that classical (distinguishable) systems use.

In 2002, Lubos Motl analytically solved the asymptotic quasinormal mode problem for all spin cases and computed their quantum frequencies ν.[9] The quantum mechanical problem of the quasinormal modes is similar to one of scattering amplitudes for a quantum particle approaching a scattering body. In this case the scattering body is the black hole; the particles either escape to infinity or are absorbed by the black hole.

Lubos noticed that the transmission amplitude $T(\omega)$, that is, the probabilities of particles escaping to infinity, is similar in form to those of statistical mechanics. For the spin-1 modes of vibration, Lubos obtained

$$T_1(\omega) \approx \frac{1}{e^{\beta_H\ \omega} - 1} \tag{23}$$

where $\beta_H = 8\pi G_N M$, is the inverse of the Hawking temperature. This is proportional to the occupation numbers for quantum objects with Bose-Einstein statistics. Similarly, for the spin-1/2 modes, Lubos obtained

$$T_{1/2}(\omega) \approx \frac{1}{e^{\beta_H\ \omega} + 1}, \tag{24}$$

proportional to the occupation numbers for Fermi-Dirac statistics. These two formulas are what would be expected for bosons with spin-1 and fermions with spin-1/2.

However, the cases of spin-0 and spin-2 did not give the expected Bose-Einstein result. The sign obtained was the "+" of Fermi-Dirac statistics instead of the "−" of Bose Einstein, and instead of the "1" appropriate to any reasonable particle, there was a "3":

$$T_0(\omega) \approx T_2(\omega) \approx \frac{1}{e^{\beta_H\ \omega} + 3}. \tag{25}$$

Lubos named the statistics appropriate for this type of occupation number "Tripled Pauli statistics" and noted that "(s)uch an occupation number can be derived for objects that satisfy the Pauli's principle, but if such an object does appear (only one of them can be present in a given state), it can appear in three different forms."

There have been no elementary particles observed with spin 0 or 2. The Higgs (spin 0) and the graviton (spin 2) are expected to have these quantum numbers. The Higgs is expected to be related to the origin of mass while the graviton is to carry the gravitational field. If the graviton is to take off the non spherical portions of the gravitational field of the black hole, these quasinormal modes of vibration are of particular interest, and they are also of interest in understanding the nature of mass.

Suppose that a particle was neither a fermion nor a boson, but satisfied Tripled Pauli statistics. In describing such a particle, we would have to include three possible forms for it. Following the convention in this paper, we will call these forms R, G, and B. Thus Tripled Pauli statistics could result in a density matrix form where the underlying group is the permutation group on 3 elements even though the particle is a point particle.

7 Conclusion

The left and right handed elementary fermions have weak quantum numbers that correspond to the Hermitian solutions to the 6×6 matrix idempotency equation for a subalgebra of the matrices that is defined by the permutation group on 3 elements. A natural conclusion is that the elementary fermions are composite particles, possibly with internal components obeying Tripled Pauli statistics.

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