

# Particle Symmetry Breaking in Density Matrix Formalism with Geometric Algebra

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The density matrix formalism, generalized by Schwinger's measurement algebra, provides a natural way of modeling distinct particles, such as electrons and neutrinos, in a single Clifford algebra. In Schwinger's formalism, the vacuum state is a "non-physical state" invented for the mathematical convenience of providing an intermediary between physical states. In this context, breaking symmetry through the mechanism of spontaneous symmetry breaking is suspect.

To provide for a naturally broken symmetry, we introduce the Particle Internal Symmetry Algebra (PISA), a generalization of Hestenes' Geometric (Clifford) Algebra. The PISA preserves the form of the Dirac equations for the particles, but breaks the symmetry by distinguishing vertices. Asymmetry is built into the relation between the tangent vectors of the spacetime manifold and the Clifford algebra. The problem of parameterizing the asymmetry, and converting between symmetric and asymmetric operator equations is solved in  $N + 1$  dimensional spacetime with either signature.

The paper is written for an audience of physicists unfamiliar with Clifford Algebra.

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## I. ARGUMENT IN BRIEF

This paper is based on simple arguments. We will give these arguments, in their entirety, in this section. While these arguments are simple, some of them are based on mathematics that are unfamiliar to many physicists, and others are based on physics that that is not included in the standard graduate classes.

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If the reader finds a particular subsection of this section to be confusing, the same arguments are made in the remaining sections of this paper at a detailed and introductory level. Subsections of this section are expanded to sections in the rest of the paper, and paragraphs of a subsection are expanded into subsections.

### A. Section 2 in brief

The first argument (second section) covers interesting things. First paragraph is about generalizing the Dirac equation.

### B. Section 3 in brief

The third section (second argument) covers what have you.

### C. Section 4 in brief

The fourth section (third argument) covers whatever.

## II. SECOND SECTION EXPANDS FIRST SUBSECTION

### A. Generalizing the Dirac Equation

The crowning achievement of standard quantum mechanics is the calculation of the  $g-2$  value for the electron to an accuracy of better than 10 decimal places. A theory claiming to be a unified field theory needs to be able to reproduce this calculation. The present theory will do this by deriving, from the principles of this theory, the

propagators and vertices used in the Feynman diagrams of the standard model.

The usual massive Dirac equation allows the movement of a single spin-1/2 fermion to be modeled in a very precise manner. On the other hand, the standard model [1] attributes fermion masses as arising from exchange of Higgs bosons. Therefore it is the massless Dirac equations that are of fundamental interest. In this sense, the various elementary fermions all share the same Dirac equation, and this makes them distinct from the various elementary bosons. This suggests that it is natural to associate a distinct, but identical, Dirac equation with each type of elementary fermion.

An obvious way of obtaining a multiple particle massless Dirac equation is to leave the gamma operators alone, but to replace the spinors, with matrices. Each column of the matrix making up the multiparticle wave can be associated with a class of elementary fermion. For example, the first column could represent the electron, the second column the red down quark, the third column the green down quark, etc. The massless Dirac equation for the electron:

$$(\gamma_\mu \partial^\mu)_{4 \times 4} (\psi_e)_{4 \times 1} = 0, \quad (1)$$

where the gamma matrices satisfy

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = g^{\mu\nu}, \quad (2)$$

can be generalized to an equation for four fermions. For example:

$$(\gamma_\mu \partial^\mu)_{4 \times 4} (\psi_e, \psi_{dR}, \psi_{dG}, \psi_{dB})_{4 \times 4} = 0, \quad (3)$$

is such a generalization. Note that this generalization is quite arbitrary in that it mixes quarks and leptons with no significant geometric distinction, but until we generalize to Clifford algebra we can use this as an example. To separate the four components of the  $4 \times 4$  wave matrix into the four different particles, one can use the projection operators:<sup>1</sup>

$$\begin{aligned} \eta_{00} &= \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} & \eta_{11} &= \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \\ \eta_{22} &= \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{pmatrix} & \eta_{33} &= \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}, \end{aligned} \quad (4)$$

where we use the notation  $\eta_{jk}$  to refer to the matrix with a one in the  $(j, k)$  location and the remainder of entries zero.

The four projection operators satisfy the following relations:

$$\begin{aligned} \eta_{jj} \eta_{kk} &= \delta_j^k \eta_{kk}, \\ \sum_k \eta_{kk} &= \hat{1}. \end{aligned} \quad (5)$$

In addition, none of these projection operators can be written as a nontrivial sum of two other projection operators. This combination of attributes defines the four projection operators as a set of “mutually annihilating primitive idempotents”.

Of course there are other sets of primitive idempotents for the  $4 \times 4$  complex matrices. For example, if  $S$  is an invertible matrix, then  $S \eta_{kk} S^{-1}$  gives another set of primitive idempotents. But the particular set given above is significant because it projects the unified wave function  $\Psi = (\psi_0, \psi_1, \psi_2, \psi_3)$  into distinct elementary fermions in a specific manner. One could then, for example, interpret the four fermions as was shown in Eq. (3). Not only does this give a geometric interpretation to the four fermions (as defined by the projection operators), but it also gives a geometric interpretation to the four components of their spinors (as defined by the projection operators when used on the left side of the spinors).

Thinking of the fermions in this manner promises to be useful in that it explains how it comes to be that linear combinations of fermions appear so often in the standard model. Therefore, to understand the symmetries of the fermions, we should examine the symmetries of the projection operators of the gamma matrices.

## B. Density Matrix Formalism

The standard model of the elementary particles is written in the traditional formalism of operators and operands. The heart of the traditional operator formalism is the concept of eigenvectors, eigenvalues and operators. For example, letting the operator be  $S_x$ , the spin operator for the  $\hat{x}$  direction, there are two eigenvectors, with eigenvalues of  $\pm 1/2$ . The eigenvector equations are:

$$\begin{aligned} S_x |+\rangle &= \frac{1}{2} \sigma_x |+\rangle = +\frac{1}{2} |+\rangle, \\ S_x |-\rangle &= \frac{1}{2} \sigma_x |-\rangle = -\frac{1}{2} |-\rangle, \end{aligned}$$

where  $\sigma_x$  is the Pauli spin matrix. In this section we will argue in favor of the density matrix  $\rho$  formalism.<sup>2</sup> In the language of bras and kets:

$$\rho_{+x} = \frac{|+x\rangle \langle +x|}{\langle +x|+x\rangle}. \quad (6)$$

<sup>1</sup> Mathematicians frequently refer to projection operators as “idempotents”.

<sup>2</sup> Density matrices are traditionally written with a hat, i.e.  $\hat{\rho}$ . We will reserve the hat for the canonical elements of a Clifford algebra as used later in this paper.

We will frequently assume that the states are normalized and leave off the denominator in the above fraction.

While the density matrix formalism is particularly convenient for mixed states, this paper will limit our analysis to the pure states usually analyzed with spinors.

The primary reason for preferring density matrix formalism to spinors is that a density matrix can be interpreted mathematically as an element of an “algebra”. An algebra is a collection of symbols with a commutative addition and a (possibly noncommutative) multiplication defined. Both operations are associative.

We can operate on kets with operators chosen from the Pauli algebra, so it is natural for us to choose the Pauli (or Dirac) algebra as the algebra that our density matrices live in. We will use the Pauli algebra as our primary example, but we will also give examples from the Dirac algebra. Later, we will generalize to the Geometric (Clifford) algebra and then the Particle Internal Symmetry Algebra.

Since the density matrix is an operator, we can consider the results when it is squared. Density matrices for pure states are projection operators - when squared they are unchanged:

$$\rho_\chi \rho_\chi = \rho_\chi, \quad (7)$$

This is an example of an idempotency relation. We will frequently use  $\iota_\chi$  to designate an idempotent corresponding to the state  $\chi$ .

An argument for preferring a density matrix formalism to spinors is that ensembles can be described in density matrix form in a particularly natural way, and there is a simple relationship between the entropy of a quantum ensemble in the density matrix form:

$$S = -k \operatorname{tr}(\iota_A \ln(\iota_A)), \quad (8)$$

where  $k$  is Boltzmann’s constant. Also, the equation of motion for the density matrix can be written in the form for Liouville’s equation in classical statistical mechanics if one modifies the Hamiltonian appropriately:

$$\partial_t \iota = -[\iota, H/i\hbar]. \quad (9)$$

A later subsection will show that the traditional Stern-Gerlach apparatus calculations are simpler in the density matrix / idempotent formulation. The primary advantage of spinors is that linear combinations of spinors are still spinors.

### C. The Schwinger Measurement Algebra

In order to unify the elementary particles, we must generalize the density matrix formalism to allow for different particle types to appear in the same algebra. The usual approach to particle unification is to find a representation of a symmetry group and to assume that the various particles transform under the symmetry as elements of

the group. The vast number of papers proposing that the particles might be unified in various such ways shows that this is a rather unrestrictive method of looking for unification.

What we will instead be doing is looking for an algebra that can represent the elementary particles among its idempotents. The number of appropriate algebras is quite small, their idempotent structures well known, so this is a very restrictive technique for representing the elementary particles. In fact, it’s a fairly obvious method and it does not work, at least without assuming that the familiar elementary particles are bound states of more fundamental particles.

The theoretical basis for an algebra that contains idempotents that correspond to distinct elementary particles was written down by Julian Schwinger.[2, Chap. 1.1] We will not follow his notation in the bulk of this paper, but present it here in abbreviated form:

Let  $a_1$  be an elementary particle in  $\mathcal{F}$ . Let  $M(a_1)$  symbolize the selective measurement that accepts particles of type  $a_1$ , and rejects all others. One can imagine some sort of Stern-Gerlach apparatus, though since we are here considering permanently bound subparticles of quarks it will have to be an imaginary apparatus. We can define addition of measurements to be the less selective measurement that accepts particles of any of the included types:

$$M(a^1) + M(a^2) = M(a^1 + a^2). \quad (10)$$

Two successive measurements can be represented by multiplication of the measurement symbols. Because of the physical interpretations of the symbols, addition is associative and commutative, while multiplication is at least associative. One and zero represent the trivial measurements that accept all or no particles.

Clearly,  $0 + M(a^1) = M(a^1)$ ,  $1M(a^1) = M(a^1)1 = M(a^1)$ , and  $0M(a^1) = M(a^1)0 = 0$ , so the set of measurements form an algebra. The “elementary” measurements associated with a complete set of fermions satisfy the following equations:

$$M(a^1)M(a^1) = M(a^1), \quad (11)$$

$$M(a^1)M(a^2) = 0, \quad a^1 \neq a^2, \quad (12)$$

$$\sum_n M(a^n) = 1 \quad (13)$$

A more general measurement process is one where the outgoing particles are different from the incoming ones. Schwinger’s notation for this process is  $M(a^1, a^2)$ , where  $a^2$  is the incoming state and  $a^1$  is the outgoing state. These sorts of measurements are defined so that there is no loss of particles in the modification of the particle type. That is:

$$M(a^1, a^2)M(a^2, a^3) = M(a^1, a^3). \quad (14)$$

In Schwinger's measurement algebra, the connection to the annihilation and creation operators of the usual quantum field theory is obtained by introducing a "fictitious" null, or vacuum state 0. From his book:

The uncontrollable disturbance attendant upon a measurement implies that the act of measurement is indivisible. That is to say, any attempt to trace the history of a system during a measurement process usually traces the nature of the measurement that is being performed. Hence, to conceive of a given selective measurement  $M(a', b')$  as a compound measurement is without physical implication. It is only of significance that the first stage selects systems in the state  $b'$ , and that the last one produces them in the state  $a'$ ; the interposed states are without meaning for the measurement as a whole. Indeed, we can even invent a non-physical state to serve as the intermediary. We shall call this mental construct the null state 0, and write

$$(2.1) \quad M(a', b') = M(a', 0)M(0, b').$$

The measurement that selects a system in the state  $b'$  and produces it in the null state,<sup>3</sup>

$$(2.2) \quad M(0, b') = \psi(b'),$$

can be described as the annihilation of a system in the state  $b'$ ; and the production of a system in the state  $a'$  following its selection from the null state,

$$M(2.3) \quad (a', 0) = \psi^\dagger(a'),$$

can be characterized as the creation of a system in the state  $a'$ . Thus the content of (2.1) is the indiscernibility of  $M(a', b')$  from the compound process of the annihilation of a system in the state  $b'$  followed by the creation of a system in the state  $a'$ ,

$$(2.4) \quad M(a', b') = \psi^\dagger(a')\psi(b') [2]$$

When Schwinger reprinted the above referenced book in 1991, he wrote a special preface with the comment: "Instead of the symbol of measurement:  $M(a', b')$ , I now write  $|a'b'\rangle$ , combining reference to what is selected and what is produced, with an indication that the act of measurement has a beginning and an end. Then, with the conceptual analysis of  $|a'b'\rangle$  into two stages, one of annihilation and one of creation, as symbolized by  $|a'b'\rangle = |a'\rangle\langle b'|$ , the fictitious null state, and the symbols  $\Psi$  and  $\Phi$  can be discarded." At that time, the use of spontaneous symmetry breaking of the vacuum was

already well established for the standard model. One wonders what he thought of it.

The "elementary" measurements that Schwinger refers to are measurements that cannot be written as a sum of (nontrivial) measurements. The corresponding concept in mathematics is "primitive" idempotents. These are idempotents that cannot be written as the sum of (nontrivial) idempotents. Our use of the Schwinger measurement algebra will be in its method of defining the elementary particles according to the spectrum of primitive idempotents.

In this paper we are generalizing the density matrix formalism to apply to distinct elementary particles. The Schwinger measurement algebra gives us the rules on how to do this, but since these are distinct formalisms, we should check to see if they are compatible. The idempotency rule, Eq. (7) gives us the rule for squaring a given state, and in this the two algebras are identical.

Since we are treating particles with different spin (and later, particles with different handedness) as distinct particles, we can also compare the two algebras when the states are distinct. Let  $\iota_A = | +z\rangle\langle +z|$  be the density matrix representation of a fermion with spin  $+1/2$  in the  $\hat{z}$  direction and let  $\iota_B = | -z\rangle\langle -z|$  be the oppositely directed state. Then the product of these two density matrices is zero:

$$\begin{aligned} \iota_A \iota_B &= | +z\rangle\langle +z| | -z\rangle\langle -z|, \\ &= | +z\rangle(\langle +z| -z\rangle)\langle -z| = 0. \end{aligned} \quad (15)$$

This is identical to the Schwinger measurement algebra rule Eq. (12). In addition, the sums over "primitive" elements of each algebra are unity. Thus the Schwinger measurement algebra gives a generalization of the density matrix representation.

We now derive some idempotency relations within the Schwinger measurement algebra for later use. Let the particles under consideration be the electron, positron, electron neutrino and anti electron neutrino, and let the orientations be defined according to some direction,  $A$ . Then the Schwinger measurement algebra is generated from the eight primitive idempotents:

$$\begin{aligned} M(eA) \quad M(\bar{e}A) \quad M(e\bar{A}) \quad M(\bar{e}\bar{A}) \\ M(\nu A) \quad M(\bar{\nu}A) \quad M(\nu\bar{A}) \quad M(\bar{\nu}\bar{A}), \end{aligned} \quad (16)$$

where, for example,  $M(\bar{\nu}\bar{A})$  is the measurement that allows the electron anti neutrino with orientation  $-A$  to pass. The full algebra includes all sums and products as well as real multiples and elements with various orientations  $A$ .

Consider one of the  $2^8 = 256$  elements<sup>4</sup> of the algebra of the form

$$\begin{aligned} \mu_\chi = \pm M(eA) \pm M(\bar{e}A) \pm M(e\bar{A}) \pm M(\bar{e}\bar{A}) \\ \pm M(\nu A) \pm M(\bar{\nu}A) \pm M(\nu\bar{A}) \pm M(\bar{\nu}\bar{A}) \end{aligned} \quad (17)$$

<sup>3</sup> Schwinger writes his annihilation and creation operators as  $\Phi(b')$  and  $\Psi(a')$ . In the interest of conforming to modern practice, I've replaced these with  $\psi(b')$  and  $\psi^\dagger(a')$ , respectively.

<sup>4</sup> Note that these 256 elements depend on the orientation  $A$

where the  $\pm$  are independent and  $\chi$  stands for the 8 values. Note that with a different choice of  $A$ , other than  $-A$ , one will obtain another 256 different  $\mu_\chi$ . Any such element, when squared, gives unity:

$$\begin{aligned}\mu_\chi^2 &= M(eA)^2 \dots + M(\bar{\nu}\bar{A})^2 + \text{Cross terms} \\ &= M(eA) + \dots + M(\bar{\nu}\bar{A}) = 1,\end{aligned}\quad (18)$$

where recourse has been taken to Eq. (12) to eliminate the cross terms, and Eq. (11) and Eq. (13) to reduce the sum of squares to one.

Since  $\mu_\chi$  squares to one, its eigenvalues are  $\pm 1$ . For example,

$$\mu_\chi M(\bar{e}A) = \pm M(\bar{e}A), \quad (19)$$

where the  $\pm$  on the right is the one appropriate to the  $\bar{e}A$  according as  $\mu_\chi$  takes a  $+$  or a  $-$  in the  $M(\bar{e}A)$  term of Eq. (17). Each  $\mu_\chi$ , naturally defines an idempotent by:

$$\begin{aligned}\iota_\chi &= \frac{1}{2}(1 + \mu_\chi), \\ &= \bar{M}(a^1) + \dots + M(a^n),\end{aligned}\quad (20)$$

where  $M(a^1), \dots, M(a^n)$  are the primitive idempotents that took  $+$  values in the  $\pm$  of Eq. (17).

The leptons listed in Eq. (16) are intriguing in that their characteristics are in binary form. For example, half of them have spin  $+1/2$  in the  $A$  direction, half of them are electrons as opposed to neutrinos, and half of them are particles rather than anti particles. We define three corresponding idempotent operators corresponding to spin in the  $A$  direction, flavor, and  $C$ :

$$\begin{aligned}\sigma_A &= +M(eA) + M(\bar{e}A) - M(e\bar{A}) - M(\bar{e}\bar{A}) \\ &\quad + M(\nu A) + M(\bar{\nu}A) - M(\nu\bar{A}) - M(\bar{\nu}\bar{A}), \\ \mu_F &= +M(eA) + M(\bar{e}A) + M(e\bar{A}) + M(\bar{e}\bar{A}) \\ &\quad - M(\nu A) - M(\bar{\nu}A) - M(\nu\bar{A}) - M(\bar{\nu}\bar{A}), \\ \mu_C &= +M(eA) - M(\bar{e}A) + M(e\bar{A}) - M(\bar{e}\bar{A}) \\ &\quad + M(\nu A) - M(\bar{\nu}A) + M(\nu\bar{A}) - M(\bar{\nu}\bar{A}).\end{aligned}\quad (21)$$

These three idempotents allow us to factor the eight primitive idempotents:

$$\begin{aligned}M(eA) &= \frac{1}{2}(1 + \sigma_A)\frac{1}{2}(1 + \mu_F)\frac{1}{2}(1 + \mu_C), \\ M(\bar{e}A) &= \frac{1}{2}(1 + \sigma_A)\frac{1}{2}(1 + \mu_F)\frac{1}{2}(1 - \mu_C), \\ M(e\bar{A}) &= \frac{1}{2}(1 - \sigma_A)\frac{1}{2}(1 + \mu_F)\frac{1}{2}(1 + \mu_C), \\ M(\bar{e}\bar{A}) &= \frac{1}{2}(1 - \sigma_A)\frac{1}{2}(1 + \mu_F)\frac{1}{2}(1 - \mu_C), \\ M(\nu A) &= \frac{1}{2}(1 + \sigma_A)\frac{1}{2}(1 - \mu_F)\frac{1}{2}(1 + \mu_C), \\ M(\bar{\nu}A) &= \frac{1}{2}(1 + \sigma_A)\frac{1}{2}(1 - \mu_F)\frac{1}{2}(1 - \mu_C), \\ M(\nu\bar{A}) &= \frac{1}{2}(1 - \sigma_A)\frac{1}{2}(1 - \mu_F)\frac{1}{2}(1 + \mu_C), \\ M(\bar{\nu}\bar{A}) &= \frac{1}{2}(1 - \sigma_A)\frac{1}{2}(1 - \mu_F)\frac{1}{2}(1 - \mu_C).\end{aligned}\quad (22)$$

There are only three non trivial operators on the right hand side, so this is a simplification.

It is sometimes useful to consider measurements that ignore the orientation of the particle. There are four such measurements:

$$\begin{aligned}M(e) &= M(eA) + M(e\bar{A}) = \frac{1}{4}(1 + \mu_F)(1 + \mu_C) \\ M(\bar{e}) &= M(\bar{e}A) + M(\bar{e}\bar{A}) = \frac{1}{4}(1 + \mu_F)(1 - \mu_C) \\ M(\nu) &= M(\nu A) + M(\nu\bar{A}) = \frac{1}{4}(1 - \mu_F)(1 + \mu_C) \\ M(\bar{\nu}) &= M(\bar{\nu}A) + M(\bar{\nu}\bar{A}) = \frac{1}{4}(1 - \mu_F)(1 - \mu_C)\end{aligned}\quad (23)$$

The above four measurements do not depend on orientation.

In his original work, Schwinger used complex phases to derive probabilities. It is traditional, in Geometric Algebra work, to eschew complex number in favor of Clifford algebraic constants that commute and happen to square to  $-1$ . For example,  $\sigma_x\sigma_y\sigma_z$  commutes with  $\sigma_A$  and squares to  $-1$  and so, for some purposes, can be interpreted as  $i$ .

It is useful to define the Schwinger measurement algebra in terms of the values that square to  $\pm 1$ , for example  $\mu_F$  and  $\mu_C$  instead of the primitive measurements such as  $M(eA)$ . The reason for doing this is first so that we can define a magnitude  $|\cdot|^2$ , and second so that we can derive relationships to Clifford algebra, Geometric algebra, and the PISA defined later in this paper.

Clearly,  $\mu_F$  and  $\mu_C$  square to unity, as does  $\sigma_x$ . These three commute as particle type commutes with orientation, as can be verified using Eq. (21). The four possible products of  $\mu_F$  and  $\mu_C$  all commute and square to unity:

$$\begin{aligned}(1)^2 &= 1 & (\mu_F)^2 &= 1 \\ (\mu_C)^2 &= 1 & (\mu_C\mu_F)^2 &= 1\end{aligned}\quad (24)$$

While  $\mu_F$  and  $\mu_C$  commute with  $\sigma_A$  for any orientation  $A$ , the Pauli spin algebra itself does not commute. Instead, they anticommute, and their products will therefore square to either  $+1$  or  $-1$ . A complete (real) basis for the Pauli algebra (of  $2 \times 2$  complex matrices) can be written as the eight elements:

$$\begin{array}{cccc} 1 & \sigma_x & \sigma_y & \sigma_z \\ \sigma_x\sigma_y & \sigma_x\sigma_z & \sigma_y\sigma_z & \sigma_x\sigma_y\sigma_z.\end{array}\quad (25)$$

Note that the elements of the top row of the above all square to  $+1$ , while the elements in the bottom row square to  $-1$ . Thus a complete basis for the Schwinger measurement algebra, in the sense of a set of basis vectors for the algebra as a real vector space, is the 32 elements given by the products of the 4 elements whose squares are given in Eq. (24) and the 8 elements shown in Eq. (25). These 32 products square to either 1 or  $-1$  according as their Eq. (25) components. Listing the 16 elements that square to  $+1$  first, the 32 products are:

$$\begin{array}{cccc} 1, & \mu_F, & \mu_C, & \mu_C\mu_F, \\ \sigma_x, & \sigma_x\mu_F, & \sigma_x\mu_C, & \sigma_x\mu_C\mu_F, \\ \sigma_y, & \sigma_y\mu_F, & \sigma_y\mu_C, & \sigma_y\mu_C\mu_F, \\ \sigma_z, & \sigma_z\mu_F, & \sigma_z\mu_C, & \sigma_z\mu_C\mu_F, \\ \sigma_x\sigma_y, & \sigma_x\sigma_y\mu_F, & \sigma_x\sigma_y\mu_C, & \sigma_x\sigma_y\mu_C\mu_F, \\ \sigma_x\sigma_z, & \sigma_x\sigma_z\mu_F, & \sigma_x\sigma_z\mu_C, & \sigma_x\sigma_z\mu_C\mu_F, \\ \sigma_y\sigma_z, & \sigma_y\sigma_z\mu_F, & \sigma_y\sigma_z\mu_C, & \sigma_y\sigma_z\mu_C\mu_F, \\ \sigma_x\sigma_y\sigma_z, & \sigma_x\sigma_y\sigma_z\mu_F, & \sigma_x\sigma_y\sigma_z\mu_C, & \sigma_x\sigma_y\sigma_z\mu_C\mu_F.\end{array}\quad (26)$$

The above 32 elements<sup>5</sup>, together with their negatives, form a group under multiplication. Any two elements

<sup>5</sup> Those familiar with Geometric algebras will note that if we are

either commute or anticommute depending only on their orientation.

Given that the Schwinger measurement algebra is thus written as a real vector space, it is natural to define a squared magnitude  $| \cdot |^2$  in the usual manner:

$$\begin{aligned} & |a_1 + a_f \mu_f + \dots + a_{xyzcf} \sigma_x \sigma_y \sigma_z \mu_C \mu_F|^2 \\ &= a_1^2 + a_f^2 + \dots + a_{xyzcf}^2, \end{aligned} \quad (27)$$

where  $a_\chi$  are real numbers.<sup>6</sup> While our definition of the squared magnitude depended on the particular orientation of the  $x$ ,  $y$  and  $z$  axes, it can be seen that a rotation of the axes leaves the magnitude unchanged. Later in this paper we will associate probabilities with  $| \cdot |^2$ .

#### D. Linearity in QM and Nature

The primary advantage of the use of spinors in quantum mechanics is that they allow wave equations to be written in linear form. For example, consider the Dirac equation:

$$\gamma^\mu \partial_\mu \psi = m \psi. \quad (28)$$

If  $\psi_1$  and  $\psi_2$  are two solutions to Eq. (28), then so is  $\alpha \psi_1 + \beta \psi_2$ , where  $\alpha$  and  $\beta$  are complex constants. From a calculational point of view, this is particularly useful in that it allows use of the Fourier transform. However, the interpretation of the linear combinations of wave functions show that it is only from a calculational perspective that quantum mechanics can be thought of as linear.

In contrast to quantum mechanics, the classical laws of electricity and magnetism are truly linear. If one possesses two solutions of Maxwell's equations, then a linear combination of them is also a solution. In this sense, classical E&M are as linear as quantum mechanics. But the two theories differ in how the linear combination is interpreted physically. In classical E&M, twice a given solution corresponds classically to a configuration of charges and currents that are twice as strong as the original configuration. By contrast, twice a quantum mechanical wave function gives a new wave function that is interpreted physically as identical to the original wave function.

In the spinor representation of QM, there is a dichotomy between normalization and linearity. The linear combination of two normalized spinor wave functions

is generally not a normalized spinor wave function. Requiring the spinor theory to be closed under linear combinations therefore requires that we included normalization factors in our calculations. For example, when computing a probability of a transition between two unnormalized spinor states we use:

$$P_{AB} = \frac{\langle A|B \rangle \langle B|A \rangle}{\langle A|A \rangle \langle B|B \rangle}. \quad (29)$$

On the other hand, if we give up the notion of linearity and arrange for our spinor wave functions to be normalized, then we can calculate probabilities in a simpler way:

$$P_{AB} = \langle A|B \rangle \langle B|A \rangle. \quad (30)$$

In the idempotent theory we have no easy way to retain a form of linearity, so there is no reason to sacrifice normality for linearity. In addition, idempotent normalization is unique; they do not possess the arbitrary complex phase that spinors do. Accordingly, we will assume that our idempotents are always normalized, and the calculation for the probability of transition is therefore:

$$\begin{aligned} P_{AB} &= \langle A|B \rangle \langle B|A \rangle, \\ &= \text{tr}(\langle A||B \rangle \langle B||A \rangle), \\ &= \text{tr}(|A \rangle \langle A||B \rangle \langle B|), \\ &= \text{tr}(\iota_A \iota_B). \end{aligned} \quad (31)$$

Note that this formula is simpler than either of the spinor formulas. The density matrix / idempotent formulation, transition probabilities are given simply by the trace of the product of the two states. We will apply this fact when computing branching ratios using the idempotent formalism.

#### E. Spin Projection Operators

In this subsection we will more carefully determine the relationship between the density matrix formalism and the spinor formalism and the spin projection operators (primitive idempotents) of the Pauli spin matrices. This is quite elementary, but is included in order to illustrate the later generalizations to Clifford algebras.

Let  $S_u$  designate the operator for spin-1/2 in the  $\hat{u} = u_x \hat{x} + u_y \hat{y} + u_z \hat{z}$  direction. In terms of the usual Pauli spin matrices, we have:

$$S_u = \sigma_u / 2 = (u_x \sigma_x + u_y \sigma_y + u_z \sigma_z) / 2. \quad (32)$$

Note that  $S_u$  squares to 1/4:

$$S_u^2 = \hat{1} / 4, \quad (33)$$

where  $\hat{1}$  is the unit operator, so that  $S^2 = S_x^2 + S_y^2 + S_z^2 = 3/4$ .

Let  $| + u \rangle$  be a ket that is an eigenvector of  $S_u$  with eigenvalue  $+1/2$ :

$$S_u | + u \rangle = (\sigma_u / 2) | + u \rangle = +\frac{1}{2} | + u \rangle. \quad (34)$$

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to associate even this simplified Schwinger measurement algebra with a Geometric algebra we will require spacetime to have at least 5 dimensions. We will therefore generalize the PISA to allow hidden dimensions beyond the usual 4.

<sup>6</sup> Those familiar with Clifford algebra will note that  $|F|^2 = \langle F^\dagger F \rangle > 0$ . An arbitrary element of the Schwinger measurement algebra can be written as  $kM(a^1, a^2)$  where  $k$  is a complex number. Then  $|F|^2$ , in Schwinger's notation, is given by  $\text{tr}(F^\dagger F)$ .

Then the corresponding density matrix is also an eigenvector of spin with the same eigenvalue:

$$S_u \rho_{+u} = S_u | +u \rangle \langle +u | = +\frac{1}{2} \rho_{+u} \quad (35)$$

The projection operator for spin in the  $\hat{u}$  direction is given by:

$$\iota_u = (\hat{1} + u_x \sigma_x + u_y \sigma_y + u_z \sigma_z) / 2 = \frac{1}{2} \hat{1} + S_u. \quad (36)$$

If  $| \rangle$  is any ket that is not annihilated by  $\iota_u$ , then  $\iota_u | \rangle$  is a ket that is an eigenvector of  $S_u$  with eigenvalue  $+\frac{1}{2}$ :

$$\begin{aligned} S_u \iota_u | \rangle &= S_u \left( \frac{1}{2} + S_u \right) | \rangle \\ &= (S_u / 2 + 1/4) | \rangle = \frac{1}{2} \left( \frac{1}{2} + S_u \right) | \rangle \\ &= \frac{1}{2} \iota_u | \rangle. \end{aligned} \quad (37)$$

Canceling the arbitrary spinor, we see that in the algebra of the Pauli spin matrices,  $\iota_u$  is an eigenvector of  $S_u$  with eigenvalue  $+\frac{1}{2}$ :

$$S_u \iota_u = +\frac{1}{2} \iota_u. \quad (38)$$

Since  $\iota_u$  and  $\rho_u$  are both elements of the Pauli algebra (i.e.  $2 \times 2$  complex matrices), and are both eigenvectors of  $S_u$  with eigenvalue  $+\frac{1}{2}$ , we suspect that they are related.

To see that  $\iota_u$  and  $\rho_u$  are identical, let us choose to define our spinor  $| +u \rangle$  in the traditional  $\hat{z}$  oriented form. If  $\hat{u} \neq \hat{z}$ , then a normalized eigenvector of

$$S_u = \frac{1}{2} \begin{pmatrix} u_z & u_x - i u_y \\ u_x + i u_y & -u_z \end{pmatrix} \quad (39)$$

is given by:

$$| +u \rangle = \frac{e^{i\alpha}}{(u_x^2 + u_y^2 + (1 - u_z)^2)^{0.5}} \begin{pmatrix} u_x - i u_y \\ 1 - u_z \end{pmatrix} \quad (40)$$

where  $\alpha$  is an arbitrary real phase that will cancel in the density matrix form:

$$\rho_u = | +u \rangle \langle +u | \quad (41)$$

$$\begin{aligned} &= \frac{1}{u_x^2 + u_y^2 + (1 - u_z)^2} \begin{pmatrix} u_x - i u_y \\ 1 - u_z \end{pmatrix} \begin{pmatrix} u_x + i u_y & 1 - u_z \end{pmatrix} \\ &= \frac{1}{2 - 2u_z} \begin{pmatrix} u_x^2 + u_y^2 & (1 - u_z)(u_x - i u_y) \\ (1 - u_z)(u_x + i u_y) & (1 - u_z)^2 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 + u_z & u_x - i u_y \\ u_x + i u_y & 1 - u_z \end{pmatrix} = \iota_u. \end{aligned} \quad (42)$$

The above calculation fails for  $\hat{u} = \hat{z}$ , but the special case can be handled by using a more suitable eigenvector in Eq. (39).

Note the similarity between the right column of Eq. (39) and Eq. (40). A little algebra will show that the

left column of Eq. (39) would have worked in a similar way. This coincidence suggests an easier way of finding eigenvectors to the spin operators, and also a method of extracting a spinor representation of a state from its idempotent representation. We will go into more detail on the method in the subsection devoted to the Stern-Gerlach calculation.

Since  $\rho_u$  is an (operator) eigenvector of  $S_u$ , then any column of  $\rho_u$  must either be a (spinor) eigenvector of  $S_u$ , or it must be zero. If we wish to pick out a particular column of  $\rho_u$ , a convenient way to do it is to zero out the remaining columns. In the example given, this could be done by either

$$\iota_{+z} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ or} \quad (43)$$

$$\iota_{-z} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (44)$$

In other words, we can obtain a spinor representation from the idempotent representation by multiplying on the right by a constant idempotent and then ignoring the degrees of freedom that have been eliminated. Thus the set of spinor states form a subalgebra of the algebra of idempotent states. The only issue is that we must be careful to avoid using a constant idempotent that annihilates the state idempotent, and even in this case we can define the result through a limiting process that we need not describe here.

From the point of view of the idempotent formulation, spinor wave functions are defined quite arbitrarily by right multiplying by an idempotent. While this gets rid of degrees of freedom that have had, little obvious use, the penalty one must pay is that one must define a quite arbitrary constant idempotent to do this. In other words, spinor representations of a state possess an arbitrariness in the form of a constant vector that is not present in the density matrix or idempotent representation. Since the density matrix representation is equivalent to the spinor representation, in terms of all possible quantum measurements, we prefer the density matrix representation.

As long as a physicist sticks to the technique of describing a quantum state with an abstract complex vector space, the problem of the implicit constant vector associated with spinor subalgebras is not manifest. However, when one attempts to write quantum mechanics using the Geometric algebra, such as was done in the groundbreaking paper by David Hestenes,[3] one ends up with the necessity of making a choice that does not correspond to any observable. In addition to being simpler, the idempotent formulation is more easily shown to be manifestly covariant.

While spinors possess fewer degrees of freedom than idempotents in that they form a subalgebra of the algebra that contains the idempotents, it is the idempotents that are uniquely defined. In the example of the Pauli spin algebra, there is exactly one idempotent that is an eigenvector of  $S_u$  with eigenvalue  $+1/2$ . Even when

spinors are normalized, there is an unphysical freedom to multiply by a complex phase as in the  $\alpha$  of Eq. (40). In the standard model, this unphysical freedom is “gauged” to define the form of the photon coupling to the fermions.

The existence of a gauge principle in the standard model is evidence in favor of the idempotent generalization of density matrices. Later in this paper we will generalize from the Pauli algebra to more complicated Clifford algebras. Upon doing this, the constant idempotents that we use to define the spinor subalgebras will also gain more complex unphysical freedom leading to the more complicated gauge bosons of the weak and strong force.

## F. The Geometric Algebra

The next subsection will make the calculations for the Stern-Gerlach experiment using the traditional spinor and the less common density matrix formalism. In addition, to prepare the reader for the transition to a general Clifford algebra, we will also calculate using a Geometric (Clifford) algebraic notation where elements of the algebra are written as sums of products of a set of canonical basis vectors. In this subsection we will introduce the canonical basis vectors.

There is much work being done on models of the elementary particles that require hidden dimensions in spacetime. For such spacetimes, the Geometric Algebra as well as the PISA are naturally defined. It is possible to produce matrix representations of these algebras, that is, generalizations of the Dirac matrices, but we will not do so in this paper. Using specific matrix representations will tend to hide the manifest covariance of the theory. In addition, if the number of hidden dimensions is odd, the resulting matrices will either have to be written with unphysical degrees of freedom (e.g.  $8 \times 8$  complex matrices with only 32 complex degrees of freedom), or they will have to be written over the quaternions. Instead of either of these unnecessarily complicating methods, we will follow the traditions of Clifford algebra and define the elements algebraically. At first, we will define the Geometric Algebra associated with standard spacetime.

We begin with a metric manifold space for example, the usual Minkowski spacetime,  $\mathcal{M}$ . This paper will restrict its attention to manifolds representing space-time so we will refer to the metric manifold space as such. Each point in  $\mathcal{M}$  is locally a copy of  $\mathcal{R}^4$ , and we assume a locally orthogonal coordinate system,  $(t, x, y, z)$ , that define the tangent space to  $\mathcal{M}$  at that point. The tangent space, as a vector space, has a basis defined by a set of four “canonical basis vectors”,  $\hat{x}, \hat{y}, \hat{z}$  and  $\hat{t}$ , pointing in the  $+$  direction indicated by the four coordinate axes.

We next use the canonical basis vectors to define a Clifford algebra associated with the given point of spacetime. The Clifford algebra is generated by defining squares and commutation relations among the canonical basis vectors. We define the squares according to the

signature of the spacetime. For the case of  $(t, x, y, z) \in \mathcal{M}$ , the signature we will use is  $-+++$ , and the Clifford algebra will be the same as the Dirac algebra. With the left column giving the Clifford algebraic notation and the right column giving the equivalent Dirac algebra notation, the squares are:

$$\begin{aligned} \hat{t}^2 &= -1 & (\gamma_0)^2 &= -\hat{1}, \\ \hat{x}^2 &= +1 & (\gamma_1)^2 &= +\hat{1}, \\ \hat{y}^2 &= +1 & (\gamma_2)^2 &= +\hat{1}, \\ \hat{z}^2 &= +1 & (\gamma_3)^2 &= +\hat{1}, \end{aligned} \quad (45)$$

where we use  $\hat{1}$  to designate the unit element (or matrix) in the Dirac algebra. While the squares are defined by the signature and different Clifford algebras may have different signatures, all Clifford algebras suppose that the canonical basis vectors anticommute. This is identical to the case of the Dirac algebra:

$$\begin{aligned} \hat{t}\hat{x} &= -\hat{x}\hat{t} & \gamma_0\gamma_1 &= -\gamma_1\gamma_0, \\ \hat{t}\hat{y} &= -\hat{y}\hat{t} & \gamma_0\gamma_2 &= -\gamma_2\gamma_0, \\ \hat{y}\hat{z} &= -\hat{z}\hat{y} & \gamma_2\gamma_3 &= -\gamma_3\gamma_2, \end{aligned} \quad (46)$$

Given these rules, it is possible to reduce any product of canonical basis vectors to a product where each canonical basis vector appears either once or not at all and in the standard order. The standard order we will use will be  $\hat{x}\hat{y}\hat{z}\hat{t}$ . Some examples, again in both the Clifford algebraic notation and in the Dirac algebra:

$$\begin{aligned} \hat{t}\hat{z}\hat{y}\hat{x} &= \hat{x}\hat{y}\hat{z}\hat{t} & \gamma_0\gamma_3\gamma_2\gamma_1 &= \gamma_1\gamma_2\gamma_3\gamma_0, \\ (\hat{y}\hat{z})(\hat{x}\hat{y}) &= -\hat{x}\hat{t} & (\gamma_2\gamma_3)(\gamma_1\gamma_2) &= -\gamma_1\gamma_3, \\ \hat{x}\hat{y}\hat{z}\hat{x}\hat{y} &= -\hat{z} & \gamma_1\gamma_2\gamma_3\gamma_1\gamma_2 &= -\gamma_3. \end{aligned} \quad (47)$$

Since a product of the four distinct canonical basis vectors of the  $\mathcal{M}$  Geometric algebra must have each canonical basis vector appear either zero or one times, there are  $2^4 = 16$  such products. These products are called “canonical basis elements”, and the canonical basis vectors are included. We will write them with their vectors listed in  $xyzzt$  order, and we will expand the hat to cover the group. The canonical basis vectors, with their Dirac algebraic equivalents, are:

$$\begin{array}{llll} 1 & \hat{1} & \widehat{yz} & \gamma_2\gamma_3 \\ \hat{x} & \gamma_1 & \widehat{yt} & \gamma_2\gamma_0 \\ \hat{y} & \gamma_2 & \widehat{zt} & \gamma_3\gamma_0 \\ \hat{z} & \gamma_3 & \widehat{xyz} & \gamma_1\gamma_2\gamma_3 \\ \hat{t} & \gamma_0 & \widehat{xyt} & \gamma_1\gamma_2\gamma_0 \\ \widehat{xy} & \gamma_1\gamma_2 & \widehat{xzt} & \gamma_1\gamma_3\gamma_0 \\ \widehat{xz} & \gamma_1\gamma_3 & \widehat{yzt} & \gamma_2\gamma_3\gamma_0 \\ \widehat{xt} & \gamma_1\gamma_0 & \widehat{xyzt} & \gamma_1\gamma_2\gamma_3\gamma_0 \end{array} \quad (48)$$

The canonical basis elements form the basis for the Clifford algebra as a vector space. That is, an element of the



(complexified<sup>7</sup>) Clifford algebra is simply a sum of canonical basis elements multiplied by real (complex) numbers. The equivalent statement in the Dirac algebra is the fact that every  $4 \times 4$  complex matrix can be written as a sum of products of the Dirac matrices. Two example of a general element of the Clifford algebra and its equivalent in the Dirac algebra are:

$$\begin{aligned} (1+i\widehat{xy})(1-\widehat{zt})/4 &\equiv (\widehat{1}+i\gamma_1\gamma_2)(\widehat{1}-\gamma_3\gamma_0)/4, \\ (3+\widehat{xt}+\widehat{t}+\widehat{xzt})/4 &\equiv (\widehat{3}+\gamma_1\gamma_0+\gamma_0+\gamma_1\gamma_3\gamma_0)/4. \end{aligned} \quad (49)$$

Note that the above two elements are idempotents, but only the first is a primitive idempotent in the Dirac algebra. In an algebra of higher dimension, neither element may be idempotent.

The square of any canonical basis element of a Clifford algebra will be either zero or one. The equivalent statement for the Dirac (or Pauli) algebra is that the square of any product of gamma matrices will be either zero or one. It will sometimes be useful to take the canonical basis elements that square to  $-1$  and multiply them by  $i$ , thereby producing a set of basis elements that all square to 1. We will likely refer to such a set of positive signature basis elements as the ‘‘canonical basis elements’’ despite their being multiplied by  $i$ .

Any two distinct canonical basis elements will either commute or anticommute. If they anticommute, then those two elements will generate a subalgebra of the Clifford algebra that is equivalent to the Pauli spin matrices. Thus  $SU(2)$  is a very common subalgebra in a Clifford algebra. If, on the other hand, the two elements anticommute, then they will generate a subalgebra equivalent to  $U(1) \times U(1)$ . It is also possible to find subalgebras that are equivalent to  $SU(3)$ , but we will have no use for them in this paper.

The Pauli spin matrices are also equivalent to a Clifford algebra. There are only four distinct products of Pauli spin matrices, so there are only 2 basis generators and the Clifford algebra cannot be placed in equivalent form to a Geometric algebra as the Dirac algebra was. The multiplication table is:

	$\widehat{1}$	$\sigma_1$	$\sigma_2$	$\sigma_3$	
$\widehat{1}$	$\widehat{1}$	$\sigma_1$	$\sigma_2$	$\sigma_3$	
$\sigma_1$	$\sigma_1$	$\widehat{1}$	$i\sigma_3$	$-i\sigma_2$	
$\sigma_2$	$\sigma_2$	$-i\sigma_3$	$\widehat{1}$	$i\sigma_1$	
$\sigma_3$	$\sigma_3$	$i\sigma_2$	$-i\sigma_1$	$\widehat{1}$	

(50)

Since  $2^2 = 4$  and there are four elements, the equivalent Clifford algebra must have 2 canonical basis vectors. All the above elements square to unity, so we will choose a signature of  $++$ . Since our Geometric algebra for  $\mathcal{M}$

contains three canonical basis elements that square to one, we can choose two of them as Pauli spin matrices. For example:

$$\begin{aligned} 1 &\equiv \widehat{1} \\ \widehat{x} &\equiv \sigma_1 \\ \widehat{y} &\equiv \sigma_2 \\ \widehat{xy} &\equiv i\sigma_3 \end{aligned} \quad (51)$$

In the above, it is clear that the  $z$  direction is being treated differently than  $x$  or  $y$ . An equivalence that treats all directions equally is obtained if we interpret  $i$  as  $\widehat{xyz}$ :

$$\begin{aligned} 1 &\equiv \widehat{1} \\ \widehat{x} &\equiv \sigma_1 \\ \widehat{y} &\equiv \sigma_2 \\ \widehat{z} &\equiv \sigma_3 \\ \widehat{xyz} &\equiv \sigma_1\sigma_2\sigma_3 = i\widehat{1} \end{aligned} \quad (52)$$

Note that  $\widehat{xyz}$ , under the rules of a Clifford algebra, squares to  $-1$  and commutes with  $\widehat{x}$ ,  $\widehat{y}$  and  $\widehat{z}$  and can therefore be treated as an equivalent of  $i$ .

The embedding of the Pauli algebra shown in Eq. (52) is only one of several that are free of a preferred direction. Using the Clifford algebra equivalent to the Dirac algebra, that is, using  $\widehat{x}$ ,  $\widehat{y}$ ,  $\widehat{z}$  and  $\widehat{t}$  as the canonical basis vectors, the natural embeddings include the following:

Pauli	A	B	C	D
$\sigma_1$	$\widehat{x}$	$\widehat{xt}$	$+i\widehat{yz}$	$+i\widehat{yzt}$
$\sigma_2$	$\widehat{y}$	$\widehat{yt}$	$-i\widehat{xz}$	$-i\widehat{xzt}$
$\sigma_3$	$\widehat{z}$	$\widehat{zt}$	$+i\widehat{xy}$	$+i\widehat{xyt}$
$i\widehat{1}$	$\widehat{xyz}$	$\widehat{xyzt}$	$i\widehat{1}$	$\widehat{t}$

(53)

These four possibilities are exhaustive as can be seen by examining the list of canonical basis elements in Eq. (48). Written in the language of the Dirac algebra, the equivalencies are:

Pauli	A	B	C	D
$\sigma_1$	$\gamma_1$	$\gamma_1\gamma_0$	$i\gamma_2\gamma_3$	$\gamma_2\gamma_3\gamma_0$
$\sigma_2$	$\gamma_2$	$\gamma_2\gamma_0$	$i\gamma_3\gamma_1$	$\gamma_3\gamma_1\gamma_0$
$\sigma_3$	$\gamma_3$	$\gamma_3\gamma_0$	$i\gamma_1\gamma_2$	$\gamma_1\gamma_2\gamma_0$
$i\widehat{1}$	$\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3\gamma_0$	$i$	$\gamma_0$

(54)

Among these four cases, the equivalent to the Pauli algebra unit imaginary,  $i\widehat{1}$ , always commutes with the equivalents to the Pauli spin matrices,  $\sigma_k$ , but only in case C does the equivalent to  $i\widehat{1}$  commute with the whole Clifford algebra. For case A,  $\widehat{xyz} \equiv \gamma_1\gamma_2\gamma_3$  commutes with only those elements that do not contain  $\widehat{t}$ . For case B,  $\widehat{xyzt} \equiv \gamma_1\gamma_2\gamma_3\gamma_0$  commutes with only those elements that contain an even number of basis elements. For case D,  $\widehat{t} \equiv \gamma_0$  commutes only with those elements that contain an even number of spatial basis elements.

Later in this paper we will generalize to allow hidden dimensions. In that case, the above list of embeddings are not exhaustive. In order to match our Geometric Algebra notation more closely, we will use  $\sigma_x$  for  $\sigma_1$ , etc,

<sup>7</sup> For most of this paper, we will be examining the complexified Clifford algebras which are simpler than the real Clifford algebras. We will often write ‘‘Clifford algebra’’ or ‘‘Geometric algebra’’ where we actually mean the complexified algebras.

as the symbols for the operators that give spin measured in the given direction. It's handy to define the three operators as a vector

$$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) = (\sigma_1, \sigma_2, \sigma_3). \quad (55)$$

Since  $\vec{\sigma}$  is a vector, we can write it as the product of a pure vector and an unoriented constant element:

$$\vec{\sigma} = (\hat{x}, \hat{y}, \hat{z}) \chi. \quad (56)$$

### G. Stern-Gerlach Filters

Following the Schwinger measurement algebra, we generalize the Stern-Gerlach apparatus to an imagined apparatus that is able to separate a beam of spin-1/2 particles according not only to spin, but also according to elementary particle type. In addition, our apparatus will only have one output, so we will call it a filter. Each Stern-Gerlach filter will select one spin and one particle type out of a beam. We will model these filters with a general Clifford algebra, but since the part of the Clifford algebra that describes the spin must be a copy of  $\mathcal{SU}(2)$ , we will use the Pauli spin matrices for that part.

We can imagine sending the output of one filter into the input of another. Later in this paper, we will need the formula for the result of applying three filters in series and we will develop that here. As with the Schwinger measurement algebra, this will be modeled by multiplication. The effect of a Stern-Gerlach filter of the type considered here is a projection operator, so our object of study will be the overall projection defined by the product of three projection operators:

$$P = P_3 P_2 P_1, \quad (57)$$

where  $P_1$  is applied first.

The projection operator associated with a given Stern-Gerlach filter corresponds to a primitive idempotent in the Schwinger formalism. The primitive idempotent can be split into two non primitive idempotents. The first projects out the desired particle type,  $\chi$ . The second projects out the desired spin,  $\vec{u}$ . These two projection operators commute:

$$P_{\chi u} = P_\chi P_u = P_u P_\chi. \quad (58)$$

In this subsection we will be primarily interested in the spin projection portion,  $P_u$  and will write it with the Pauli notation:

$$P_u = \frac{1}{2}(1 + \sigma_u) = \frac{1}{2}(1 + \vec{u} \cdot \vec{\sigma}). \quad (59)$$

The fundamental multiplication rule for spin operators of the Pauli algebra can be written as:

$$\sigma_u \sigma_v = \vec{u} \cdot \vec{v} + i(\vec{u} \times \vec{v}) \cdot \vec{\sigma}. \quad (60)$$

In the previous subsection we embedded the Pauli algebra in the Dirac algebra in four natural ways. Thus the  $i$  of

the Pauli algebra, as used for brevity in this subsection, can be more generally replaced with the product:

$$\hat{i} = \sigma_x \sigma_y \sigma_z. \quad (61)$$

It may be noted that while  $\hat{i}$  commutes with  $\sigma_\chi$ , it does not necessarily commute with all elements of the Geometric algebra. Since we will be dealing only with  $\vec{\sigma}$  in this subsection, this possibility does not yet concern us.

Our analysis will apply to sequences of any number of Stern-Gerlach filters, but the full results will be clear from considering just three. Accordingly, let  $\vec{w}$ ,  $\vec{v}$  and  $\vec{u}$  be three unit vectors giving the spin direction of, respectively, the first, second and third Stern-Gerlach filters, with each filter passing only particles of type  $\chi$ . The three projection operators can be written as:

$$\begin{aligned} P_{\chi u} &= P_\chi \frac{1}{2}(1 + \sigma_u), \\ P_{\chi v} &= P_\chi \frac{1}{2}(1 + \sigma_v), \\ P_{\chi w} &= P_\chi \frac{1}{2}(1 + \sigma_w). \end{aligned} \quad (62)$$

With the full operator for the sequence of filters given by the product. Since this product is not in general a projection operator, we will refer to it as a filter,  $F_{SG}$ :

$$F_{SG} = P_{\chi u} P_{\chi v} P_{\chi w}. \quad (63)$$

We first simplify the product of these three primitive idempotents by factoring out the particle projection operators:

$$\begin{aligned} F_{SG} &= P_{\chi u} P_{\chi v} P_{\chi w} \\ &= (P_\chi P_u) (P_\chi P_v) (P_\chi P_w) \\ &= (P_\chi P_\chi P_\chi) (P_u P_v P_w) \\ &= P_\chi (P_u P_v P_w). \end{aligned} \quad (64)$$

Thus we can restrict our attention to the problem of multiplying three spin projection operators:  $P_u P_v P_w$ .

In order to avoid a later division by zero, we will assume that  $\vec{u}$  and  $\vec{w}$  are not antiparallel. That is, we will assume that  $\vec{u} \cdot \vec{w} > -1$ . Under this assumption, it can be shown that there is a complex constant  $\Gamma_{uvw}$  that depends on  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  and that satisfies:

$$P_u P_v P_w = \Gamma_{uvw} P_u P_w \quad (65)$$

The existence of this constant is equivalent to noting that Stern-Gerlach filters obliterate all knowledge of previous measurements. Since the  $P_v$  measurement is hidden from the exterior of the sequence of filters by  $P_u$  and  $P_w$ , the only effect it can contribute is a change in the amplitude or phase or both.

The above reduction rule Eq. (65) allows us to reduce almost any product of spin projection operators to a complex constant multiplied by the leading and trailing operators. For example:

$$\begin{aligned} P_a P_b P_c P_d P_e &= \Gamma_{abc} (P_a P_c) P_d P_e \\ &= \Gamma_{abc} \Gamma_{acd} \Gamma_{ade} P_a P_e \end{aligned} \quad (66)$$

We may still be able to evaluate cases with antiparallel operators if we can arrange to eliminate the middle

projection operators in an order that avoids having two antiparallel operators adjacent.

A convenient way to evaluate  $\Gamma_{uvw}$  is with traces, but since this section does double duty in introducing Clifford algebraic notation, we will instead introduce “blade” notation. Blades are vector subspaces of a real Clifford algebra. The scalars of a real Clifford algebra form the “0-blade”, which are written  $\langle \rangle_0$ . The vectors form the “1-blade”. Bivectors form the “2-blade”, etc. Given a real Clifford algebraic constant, the  $n$ -blade (i.e.  $\langle \rangle_n$ ) portion can be extracted from it. If  $\kappa$  is a Clifford algebraic constant, it can be written as a sum over its blade components so:

$$\kappa = \sum_n \langle \kappa \rangle_n \quad (67)$$

Blades are defined for real Clifford algebras. For the example of the Pauli algebra, we get the real algebra by replacing  $i$  with  $\sigma_x \sigma_y \sigma_z$ . Let  $M$  be an arbitrary element of the real Pauli algebra:

$$M = a_0 + a_x \sigma_x + a_y \sigma_y + a_z \sigma_z + a_{ix} \sigma_y \sigma_z + a_{iy} \sigma_z \sigma_x + a_{iz} \sigma_x \sigma_y + a_i \sigma_x \sigma_y \sigma_z, \quad (68)$$

where  $a_\chi$  are real constants. Then the four blades are:

$$\begin{aligned} \langle M \rangle_0 &= a_0 \\ \langle M \rangle_1 &= a_x \sigma_x + a_y \sigma_y + a_z \sigma_z \\ \langle M \rangle_2 &= a_{ix} \sigma_y \sigma_z + a_{iy} \sigma_z \sigma_x + a_{iz} \sigma_x \sigma_y \\ \langle M \rangle_3 &= a_i \sigma_x \sigma_y \sigma_z. \end{aligned} \quad (69)$$

Written in the canonical basis of the Clifford algebra  $\mathcal{CL}(3,0)$ , the four blades are:

$$\begin{aligned} \langle M \rangle_0 &= a_0 \\ \langle M \rangle_1 &= a_x \hat{x} + a_y \hat{y} + a_z \hat{z} \\ \langle M \rangle_2 &= a_{ix} \widehat{yz} - a_{iy} \widehat{xz} + a_{iz} \widehat{xy} \\ \langle M \rangle_3 &= a_i \widehat{xyz}. \end{aligned} \quad (70)$$

Since  $\Gamma_{uvw}$  is a complex valued function, it will be convenient to use the complex Pauli algebra. There are then two blades, which we will denote by  $[ ]_n$  so that we can switch between the real and complex representations of  $SU(2)$ :

$$\begin{aligned} [M]_0 &= \langle M \rangle_0 + i \langle M \rangle_3 = a_0 + i a_i \\ [M]_1 &= \langle M \rangle_1 + i \langle M \rangle_2. \end{aligned} \quad (71)$$

When the Pauli algebra is represented in  $2 \times 2$  complex matrices,  $[M]_0 = \text{tr}(M)/2$ . For example,

$$\left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right]_0 = \frac{a+d}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (72)$$

In this paper we will not necessarily restrict ourselves to irreducible representations, so this relationship is not general.

We can find  $\gamma_{uvw}$  by examining the complex scalar portions of Eq. (65):

$$[P_u P_v P_w]_0 = \Gamma_{uvw} [P_u P_w]_0. \quad (73)$$

Using Eq. (60) and dividing by  $[P_u P_w]_0$  gives:

$$\Gamma_{uvw} = \frac{1}{2} + \frac{\vec{u} + \vec{w} + i\vec{w} \times \vec{u}}{2(1 + \vec{u} \cdot \vec{w})} \cdot \vec{v}. \quad (74)$$

Thus  $\Gamma_{uvw}$ , as a function of  $\vec{v}$  consists of a constant  $1/2$ , and the dot product of  $\vec{v}$  with a complex vector that depends on  $\vec{u}$  and  $\vec{w}$ .<sup>8</sup>

## H. Stern-Gerlach Interference

The presence of the constant  $i$  in Eq. (74) is interesting in that it shows that we can arrange for interference using only Stern-Gerlach filters. Since this paper is directed at a zitterbewegung model of subparticles, where the subparticles can convert from one spin state to another, the presence of a possible interference is important. This tells us that when a composite particle made up of zitterbewegung subparticles changes its spin direction, their is opportunity for interference due to spin interactions.

When we passed from the wave function (spinor) formalism to the density matrix formalism, we eliminated the  $\mathcal{U}(1)$  gauge freedom. Nor is this interference a wave effect that depends on path as in the two slit experiment. Thus this interference is a physical effect associated with the Clifford algebra in which the spin  $SU(2)$  is embedded.

In order to better understand the interference effect that can be ascribed to Stern-Gerlach filters, let us consider a spin-1/2 fermion particle beam split by a beam splitter into two beams that pass through similar Stern-Gerlach composite filters. The two filters have identical first and last spin projection operators, but the central projection operators are different. After the beams pass through the Stern-Gerlach filters, they are recombined as shown in Fig. (1). The middle filter effects the interference pattern as we now calculate.

Ignoring the operation of the beam splitter, the operator  $F_{IE}$  for the interference experiment is given by the sum of two products of spin operators:

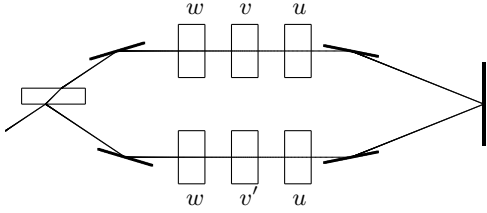
$$F_{IE} = P_u P_v P_w + P_u P_{v'} P_w. \quad (75)$$

Applying Eq. (74) we can see the interference term:

$$\begin{aligned} F_{IE} &= \left( \frac{1}{2} + \frac{\vec{u} + \vec{w} + i\vec{w} \times \vec{u}}{2(1 + \vec{u} \cdot \vec{w})} \cdot \vec{v} \right) P_u P_w \\ &+ \left( \frac{1}{2} + \frac{\vec{u} + \vec{w} + i\vec{w} \times \vec{u}}{2(1 + \vec{u} \cdot \vec{w})} \cdot \vec{v}' \right) P_u P_w \\ &= \left( 1 + \frac{\vec{u} + \vec{w} + i\vec{w} \times \vec{u}}{2(1 + \vec{u} \cdot \vec{w})} \cdot (\vec{v} + \vec{v}') \right) P_u P_w \end{aligned} \quad (76)$$

<sup>8</sup> The possible division by zero in Eq. (74) could be avoided by using the more general Schwinger measurement algebra used in Eq. (14), but those measurement algebra symbols are not as simple to translate into a Clifford algebra.

FIG. 1: A beam of spin-1/2 fermions begins at the left and is split into two beams by a beam splitter. The two beams are each passed through a series of 3 Stern-Gerlach filters. The first and last filter of each series are identical, but the central filter differs. After the series of filters, the two beams are brought together to generate an interference pattern. The orientation of the center filters effects the intensity of the beam and also produces interference effects detectable only in the interference pattern.



If  $\vec{v} = \vec{v}'$ , then while there will still be a complex rotation, there will be no interference between the two beams. The two different directions  $\vec{v}$  and  $\vec{v}'$  create two different paths from  $\vec{u}$  to  $\vec{w}$ . The interference between these two different paths turns out to be proportional to the surface area of the quadrilateral on the unit sphere defined by the four points  $\vec{u}, \vec{v}, \vec{w}, \vec{v}'$ . To show this, it is useful to consider closed paths, that is, to consider sequences of Stern-Gerlach filters that begin and end with the same orientation.

If  $\vec{A} = \vec{B}$ , Eq. (74) reduces to:

$$\Gamma_{ABA} = \frac{1 + \vec{A} \cdot \vec{B}}{2}. \quad (77)$$

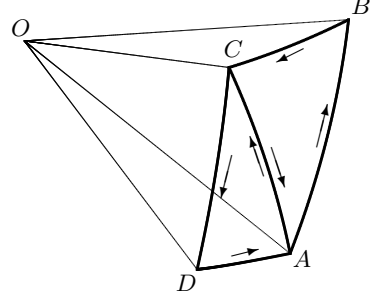
and  $P_u P_w$  reduces to  $P_A P_A = P_A$ . This shows that when we consider only a single Stern-Gerlach filter placed between two identical Stern-Gerlach filters, the result is equivalent to a single lossy Stern-Gerlach filter. While these are closed paths, they enclose no surface area, so the change in phase is zero.

Let's consider the closed spin projection paths around two adjoining spherical triangles on the surface of the unit sphere:  $\triangle ABC$ ,  $\triangle ACD$  and  $\diamond ABCD$  as shown in Fig. (2). There are three closed paths that all start and end at  $A$  (and have the same orientation). As an operator, each is associated with a complex multiple of  $P_A$ :

$$\begin{aligned} F_{ABC} &= P_A P_B P_C P_A &= \Gamma_{ABC} P_A \\ F_{ACD} &= P_A P_C P_D P_A &= \Gamma_{ACD} P_A \\ F_{ABCD} &= P_A P_B P_C P_D P_A &= \Gamma_{ABC} P_A P_C P_D P_A \\ & &= \Gamma_{ABC} \Gamma_{ACD} P_A \end{aligned} \quad (78)$$

The complex multiple of  $P_A$  associated with  $\diamond ABCD$  is the product of the complex multiples of  $P_A$  associated

FIG. 2: Two adjoining spherical triangles on the unit sphere. The complex phase parts of  $\Gamma_{uvw}$  are additive, that is,  $\arg(\Gamma_{ABCD}) = \arg(\Gamma_{ABC}) + \arg(\Gamma_{ACD})$ , and therefore the complex phase of a series of projection operators can be determined by the spherical angle they encompass.



with  $\triangle ABC$  and  $\triangle ACD$ . Thus the phases are related:

$$\arg(\Gamma_{ABC} \Gamma_{ACD}) = \arg(\Gamma_{ABC}) + \arg(\Gamma_{ACD}). \quad (79)$$

The argument can be extended to arbitrary shapes, so the phase of a closed oriented path of spin projection operators must be proportional to the surface area bounded by the path. The constant of proportionality can be found by looking at infinitesimal paths and is  $1/2$ . The resulting formula for the phase of a product of spin projection operators is:

$$\begin{aligned} &\arg(\Gamma_{ABC} \Gamma_{ACD} \Gamma_{ADE} \dots) \\ &= \arg(\Gamma_{ABC}) + \arg(\Gamma_{ACD}) + \arg(\Gamma_{ACE}) + \dots \\ &= \frac{1}{2} (\text{Spherical area defined by path } ABCDE \dots), \end{aligned} \quad (80)$$

Thus there is a geometric relationship between phase changes of spinors and paths through which they are rotated.

If we consider a path defined by spin operators that goes around the equator of the unit sphere, the surface area cut by the path will be  $4\pi/2 = 2\pi$ , and the complex phase associated will be  $2\pi/2 = \pi$ . For example, let the sequence of orientation vectors be  $\hat{z}, \hat{x}, -\hat{z}, -\hat{x}$  and  $\hat{z}$ . The operator for this sequence of Stern-Gerlach filters is:

$$\begin{aligned} F &= P_z P_x P_{-z} P_{-x} P_z \\ &= \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= -\frac{1}{4} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = -\frac{1}{4} P_z \end{aligned} \quad (81)$$

The overall effect of the above filter is to multiply the  $S_z = +1/2$  component of a spinor by  $-1/4$  and to annihilate the  $S_z = -1/2$  component. Thus the amplitude will decrease by a factor of 16 and a phase rotation of  $\pi$  will be effected.

This is an example of the well known fact that a spinor, when rotated through an angle of  $2\pi$ , gets multiplied by

–1. First, it should be noted that the multiplication by –1 only occurs if the spinor is rotated about an axis that is perpendicular to the spinor. The surface area of the sphere that a vector cuts when it is rotated about an angle of  $2\pi$  when the axis of rotation is at an angle of  $\theta$  with respect to the vector is

$$S = \int_0^\theta 2\pi \sin(\theta) d\theta = 2\pi(1 - \cos(\theta)). \quad (82)$$

Thus a spinor rotated through  $2\pi$  about an angle  $\theta$  from the axis of the spinor will be transformed as:

$$|\chi\rangle \rightarrow e^{i\pi(1-\cos(\theta))}|\chi\rangle. \quad (83)$$

When  $\theta = \pi/2$ , the above formula gives the usual negation of a spinor. The same formula gives the result of rotating a spin projection operator.

### I. Magnitudes and Probabilities

Let us consider a complex Stern-Gerlach filter,  $F$ , that begins and ends with simple filter oriented in the  $A$  direction that accepts only electrons. Using Eq. (74), we see that the complex filter will be given by a complex multiple of a simple filter oriented in the  $A$  direction. Using  $i = \sigma_x \sigma_y \sigma_z$ , we can rewrite the complex constant into elements of the Schwinger measurement algebra. Letting  $k_R$  and  $k_I$  be the real constants for the real and imaginary parts of the complex number, we can write the overall effect of the complex Stern-Gerlach filter as:

$$F = P_A P_B P_C \dots P_A = (k_R + k_I \sigma_x \sigma_y \sigma_z) P_A. \quad (84)$$

Suppose that we apply a beam of electrons of orientation  $A$  to the  $F$  filter. Using standard quantum mechanics, we know that the beam will be multiplied by the complex constant

$$\kappa = \kappa_R + i\kappa_I, \quad (85)$$

and that the probability of an electron of the beam making it through the complex filter will be:

$$\text{Prob}(e_A \text{ pass } F) = k_R^2 + k_I^2. \quad (86)$$

It should be noted that this probability depends on our applying a beam with the correct orientation to the filter.

A more fundamental probability would be the probability of a particle from a completely unpolarized beam<sup>9</sup> passing the complex filter  $F$ . To compute that probability, we have to take into account the absorption of the particles that do not have the correct orientation and / or particle type. Since there are 2 orientations and 4

particle types, the probability of a particle having the correct orientation and particle type is  $1/(2 * 4) = 1/8$ . The overall probability for the complex filter is therefore:

$$\text{Prob}(\text{pass } F) = (k_R^2 + k_I^2)/8. \quad (87)$$

This is identical to  $|F|^2$  as we now show by direct computation. Since  $| \quad |^2$  is symmetric under rotations, we can assume that  $A = \hat{z}$ . We can then easily expand  $F$  in the basis given by Eq. (26) and then apply Eq. (27):

$$\begin{aligned} |F|^2 &= |(k_R + k_I \sigma_x \sigma_y \sigma_z)(1 + \sigma_z)(1 + \mu_F)(1 + \mu_C)/8|^2 \\ &= |k_R(1 + \mu_C + \mu_F + \mu_C \mu_F \\ &\quad + \sigma_z + \sigma_z \mu_C + \sigma_z \mu_F + \sigma_z \mu_C \mu_F) \\ &\quad + k_I(\sigma_x \sigma_y \sigma_z + \sigma_x \sigma_y \sigma_z \mu_C + \sigma_x \sigma_y \sigma_z \mu_F \\ &\quad + \sigma_x \sigma_y \sigma_z \mu_C \mu_F + \sigma_x \sigma_y + \sigma_x \sigma_y \mu_C \\ &\quad + \sigma_x \sigma_y \mu_F + \sigma_x \sigma_y \mu_C \mu_F)|^2/64 \\ &= (8k_R^2 + 8k_I^2)/64 = (k_R^2 + k_I^2)/8. \end{aligned} \quad (88)$$

This gives us a definition of the probability that is purely defined within the Schwinger measurement algebra:

$$P(\text{pass } F) = |F|^2. \quad (89)$$

This definition of probability also works for complex Stern-Gerlach filters where the input state is different from the output state. For example, let  $\hat{u}$  and  $\hat{v}$  be two unit vectors,  $\theta$  the angle between them, and compute the squared magnitude of the product of  $P_u$  and  $P_v$ :

$$\begin{aligned} |P_u P_v|^2 &= |(1 + \sigma_u)(1 + \sigma_v)/4|^2 \\ &= |1 + \sigma_u + \sigma_v + \vec{u} \cdot \vec{v} + i(\vec{u} \times \vec{v}) \cdot \vec{\sigma}|^2/16 \\ &= |(1 + \vec{u} \cdot \vec{v}) + (u_x + v_x)\sigma_x + (u_y + v_y)\sigma_y + (u_z + v_z)\sigma_z \\ &\quad + ([\vec{u} \times \vec{v}]_x)\sigma_y \sigma_z + ([\vec{u} \times \vec{v}]_y)\sigma_z \sigma_x + ([\vec{u} \times \vec{v}]_z)\sigma_x \sigma_y|^2/16 \\ &= [(1 + \vec{u} \cdot \vec{v})^2 + (u_x + v_x)^2 + (u_y + v_y)^2 \\ &\quad + (u_z + v_z)^2 + |\vec{u} \times \vec{v}|^2]/16 \\ &= (1 + 2 \cos(\theta) + \cos^2(\theta) + 1 + 2 \cos(\theta) + 1 + \sin^2(\theta))/16 \\ &= (1 + \cos(\theta))/4. \end{aligned} \quad (90)$$

For incoming particles with orientation  $\vec{v}$ , the probability of passage is just twice this, or the familiar  $(1 + \cos(\theta))/2$ . Note that this calculation is much simpler than the corresponding calculation using spinors in that it avoids any necessity for solving eigenvector equations, and has no need for choosing an arbitrary representation in matrices. In addition, if one passes to a Geometric Algebra, it can be put into a purely geometric form with no need for complex numbers.

If we geometrized the Schwinger measurement algebra by going directly to the Geometric algebra, we would have no way of distinguishing between the electron and neutrino. Instead, we will slightly generalize the Geometric algebra, call the result the Particle Internal Symmetry Algebra, and show that this allows a broken symmetry that matches the  $SU(2)$  structure seen in that lepton family.

<sup>9</sup> unpolarized in the sense of particle type as well as particle orientation

## J. The Particle Internal Symmetry Algebra

In terms of Feynman diagrams, the electron and the neutrino use the same propagator. Where they differ is in their interaction vertices. In order to break the symmetry between electron and neutrino we need to break the symmetry of their vertices without breaking the symmetry of their propagators. Vertices are associated with probabilities, so if our modification of the Geometric algebra modifies the  $|\cdot|^2$  it may break the symmetry in a way that is useful.

The propagators are solutions to the usual (massless) Dirac equation:

$$(\hat{t}\partial_t - c(\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z))\psi = \nabla\psi = 0. \quad (91)$$

With a generalized Dirac equation as is natural for a Geometric algebra, the usual Dirac equations are obtained by right multiplication with idempotents such as the  $\eta_{jj}$  of Eq. (4). To preserve the idempotent structure we will require that the new algebra also be a Clifford algebra.

In the definition of the Geometric algebra, the only freedom left is the requirement that the tangent vectors be associated with the canonical basis vectors of the associated Clifford algebra. We can loosen this association by allowing the tangent vectors to be associated with more arbitrary elements of the Clifford algebra. Such a modification will transform the Clifford algebraic elements  $\hat{t}$ ,  $\hat{x}$ ,  $\hat{y}$ , and  $\hat{z}$ , of Eq. (91) but will leave unchanged the tangent vectors  $\partial_t$ ,  $\partial_x$ ,  $\partial_y$  and  $\partial_z$ .

Suppose that  $c_\alpha$  is a Clifford algebraic constant that satisfies the following commutation relations<sup>10</sup>:

$$\begin{aligned} \hat{t}c_\alpha &= c_\alpha\hat{t}, \\ \hat{x}c_\alpha &= c_\alpha^{-1}\hat{x}, \\ \hat{y}c_\alpha &= c_\alpha^{-1}\hat{y}, \\ \hat{z}c_\alpha &= c_\alpha^{-1}\hat{z}, \dots \end{aligned} \quad (92)$$

where ... signifies that we require  $c_\alpha$  to anticommute with any other spatial canonical vectors (in the case of hidden spatial dimensions).

Under these restrictions, it can be verified that the transformation that replaces the four canonical basis vectors is as follows:

$$\begin{aligned} \text{GA} &\rightarrow \text{PISA} \\ \hat{t} &\rightarrow \hat{t}, \\ \hat{x} &\rightarrow c_\alpha\hat{x}, \\ \hat{y} &\rightarrow c_\alpha\hat{y}, \\ \hat{z} &\rightarrow c_\alpha\hat{z}, \end{aligned} \quad (93)$$

which preserves the addition and multiplication relations of the Clifford algebra. For example,

$$\begin{aligned} (c_\alpha\hat{x})^2 &= c_\alpha\hat{x}c_\alpha\hat{x} \\ &= c_\alpha c_\alpha^{-1}\hat{x}^2 \\ &= \hat{x}^2 \\ (c_\alpha\hat{x})(c_\alpha\hat{y}) &= c_\alpha c_\alpha^{-1}\hat{x}\hat{y} \\ &= -c_\alpha c_\alpha^{-1}\hat{y}\hat{x} \\ &= -(c_\alpha\hat{y})(c_\alpha\hat{x}) \\ (c_\alpha\hat{x})\hat{t} &= -c_\alpha\hat{t}\hat{x} \\ &= -\hat{t}(c_\alpha\hat{x}). \end{aligned} \quad (94)$$

Therefore, the asymmetric Dirac equation:

$$(\hat{t}\partial_t - c_\alpha(\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z))\psi = 0, \quad (95)$$

can in no way be distinguished from the usual symmetric one Eq. (91). Since the Clifford algebras associated with their respective tangent vectors are identical, the two equations correspond to identical sets of coupled differential equations. All we have changed is the names of the effective basis vectors. And of course the idempotent structures of the two Clifford algebras are also identical so the spinor structures are the same.

While the Dirac equation has been preserved unchanged, the squared magnitude  $|\cdot|^2$  given in Eq. (27) cannot be written in terms of Clifford addition and multiplication and so will not, in general, be preserved as we now show. Let  $c_\alpha = 0.8 + 0.6\hat{t}$ . Then  $c_\alpha^{-1} = 0.8 - 0.6\hat{t}$  and Eq. (92) is satisfied. The transformation from Geometric algebra to PISA changes the squared magnitude of the element  $\hat{t} + \hat{x}\hat{t}$  as follows:

$$\begin{aligned} \text{GA} & & \text{PISA} \\ \hat{t} + \hat{x}\hat{t} &\rightarrow (1 - 0.6)\hat{t} + 0.8\hat{x}\hat{t} \\ |\dots|^2 = 2.0 &\rightarrow |\dots|^2 = 0.16 + 0.64 = 0.8 \end{aligned} \quad (96)$$

Using this same value for  $c_\alpha = 0.8 + 0.6\hat{t}$ , the reader can verify the following transformations of squared magnitudes:

GA	$ \text{GA} ^2$	$ \text{PISA} ^2$
$\hat{t}$	1.00	1.00
$\hat{x}$	1.00	1.00
$\hat{x}\hat{t}$	1.00	1.00
$\hat{t} + \hat{x}$	2.00	2.00
$\hat{t} - \hat{x}$	2.00	2.00
$\hat{t} + \hat{x}\hat{t}$	2.00	0.80
$\hat{t} - \hat{x}\hat{t}$	2.00	3.20

(97)

The PISA will give different squared magnitudes for elements that are written as sums and differences of canonical basis elements, provided the two canonical basis elements being summed are mixed by  $c_\alpha$ . It turns out that this will be exactly the symmetry breaking needed to distinguish the electrons and neutrinos on the basis of weak isospin  $t_3$  as we will show later in this paper.

As an alternative interpretation, the reader can consider the transformation as leaving the Geometric Algebra unmodified, but changing the squared magnitude. In

<sup>10</sup> The commutation relations, in addition to allowing a modification of the Clifford algebra canonical basis vectors in a way that preserves the Clifford algebra structure, also happen to be precisely the modifications that can be made to the Dirac equation spatial components so as to retain it as a square root of the unmodified Klein-Gordon equation. This is how the author first found these relations as well as the parameterization shown in the next subsection.[4]

fact, our calculations will be consistent with this interpretation. After finishing the details of parameterization in the following subsection, when we use the Dirac equation it will be with  $c = 1$  and all our eigenvector and eigenvalue equations will be done in the Geometric Algebra. It is only when we need a probability that we will convert to the PISA algebra.

Up to now, all our work has been with the Geometric algebra or the Schwinger measurement algebra. We need some tools to allow conversion between solutions to Geometric algebra and PISA wave equations. Such a tool will allow us to make otherwise difficult PISA computations using easier Geometric algebra techniques.

Let  $\psi_S$  be a solution to the usual Dirac equation Eq. (91). Assuming that  $c_\alpha$  has a square root (it does, as we show in the next subsection) and that the square root satisfies the same commutation relations given in Eq. (92). We can calculate as follows:

$$\begin{aligned} (\hat{t}\partial_t - (\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z + \hat{s}\partial_s))\psi &= 0, \\ c_\alpha^{0.5}(\hat{t}\partial_t - (\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z + \hat{s}\partial_s))c_\alpha^{-0.5}c_\alpha^{0.5}\psi &= 0, \\ (\hat{t}\partial_t - c_\alpha^{0.5}c_\alpha^{0.5}(\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z + \hat{s}\partial_s))(c_\alpha^{0.5}\psi) &= 0, \\ (\hat{t}\partial_t - c_\alpha(\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z + \hat{s}\partial_s))(c_\alpha^{0.5}\psi c_\alpha^{-0.5}) &= 0. \end{aligned}$$

Therefore, if  $\psi_S$  is a solution of the usual Dirac equation, then a solution to the PISA Dirac equation is given by:

$$\psi_A = c_\alpha^{0.5}\psi_S c_\alpha^{-0.5}. \quad (98)$$

If  $\psi_S$  is written as a primitive idempotent, for example:

$$\psi_S = (1 + \sigma_z)(1 + \mu_F)(1 - \mu_C)/8, \quad (99)$$

then  $\psi_A$  will also be a primitive idempotent, but with modified operators:

$$\psi_S = (1 + c_\alpha^{0.5}\sigma_z c_\alpha^{-0.5})(1 + c_\alpha^{0.5}\mu_F c_\alpha^{-0.5})(1 - c_\alpha^{0.5}\mu_C c_\alpha^{-0.5})/8. \quad (100)$$

The modified primitive idempotent, operated on by the modified operator, will possess the same quantum numbers that the unmodified idempotent had when operated on by the unmodified operator. For example:

$$c_\alpha^{0.5}\mu_F c_\alpha^{-0.5}(1 - \mu_F) = -(1 - \mu_F), \quad (101)$$

Thus our quantum numbers will be unchanged by the transformation.

If an operator has an even number of spatial components, then it will be unmodified in the conversion from GA to PISA. Otherwise, it will be multiplied on the left by  $c_\alpha$ . For example:

$$\begin{aligned} c_\alpha^{0.5}\hat{t}c_\alpha^{-0.5} &= \hat{t} \\ c_\alpha^{0.5}\hat{x}c_\alpha^{-0.5} &= c_\alpha\hat{x} \\ c_\alpha^{0.5}\hat{xt}c_\alpha^{-0.5} &= c_\alpha\hat{xt} \\ c_\alpha^{0.5}\widehat{xy}c_\alpha^{-0.5} &= \widehat{xy} \\ c_\alpha^{0.5}\widehat{xyt}c_\alpha^{-0.5} &= \widehat{xyt} \\ c_\alpha^{0.5}\widehat{xyz}c_\alpha^{-0.5} &= c_\alpha\widehat{xyz} \end{aligned} \quad (102)$$

In the next subsection, we will derive the solutions to the  $c_\alpha$  commutation relations Eq. (92) and parameterize these solutions in a convenient way.

## K. PISA Parameterization

In a real Geometric algebra, the choice of signature is significant. Up to now, this paper has restricted itself to a signature of  $(-+++)$ . For this section, we will generalize to either signature and will keep track of them with  $\pm$  as follows:

$$\begin{aligned} \hat{t}^2 &= \mp 1, \\ \hat{x}^2 &= \pm 1, \end{aligned} \quad (103)$$

so that the upper sign gives our usual signature  $(-+++)$  and the lower sign gives the opposite signature  $(+---)$ . In addition, we will generalize the number of spatial dimensions from 3 to  $M$ . For ease in computation, we will designate the  $M$  spatial basis vectors as  $\hat{x}, \widehat{x_2}, \dots, \widehat{x_M}$ . A useful element is the product of the spatial vectors which we will call  $\hat{p}$ :

$$\hat{p} = \widehat{x_2 x_3 \dots x_M}. \quad (104)$$

A useful table of squares is:

$$\begin{array}{l} \text{if } M \bmod(4) \equiv 0, 1, 2, 3 \\ \text{then } \hat{x}^2 = \pm \pm \pm \pm 1 \\ \hat{t}^2 = \mp \mp \mp \mp 1 \\ \hat{xt}^2 = + + + + 1 \\ \hat{p}^2 = + \pm - \mp 1 \\ \widehat{xp}^2 = \mp + \pm - 1 \\ \widehat{pt}^2 = \mp + \pm - 1 \\ \widehat{xpt}^2 = - \mp + \pm 1 \end{array} \quad (105)$$

It's useful to have a table of commutation relations for  $\hat{x}$ ,  $\hat{p}$  and  $\hat{t}$  so that they may be reordered:

$$\begin{array}{l} \text{if } M \bmod(4) \equiv 0, 1, 2, 3 \\ \text{then } \hat{t}\hat{p} = + - + - \widehat{pt} \\ \hat{p}\hat{x} = - + - + \widehat{xp} \\ \hat{t}\hat{x} = - - - - \widehat{xt}. \end{array} \quad (106)$$

We will use the following anticommutators (i.e.  $\{A, B\} = AB + BA$ ):

$$\begin{array}{l} \text{if } M \bmod(4) \equiv 0, 1, 2, 3 \\ \text{then } \{\hat{x}, \widehat{xt}\} = 0 \ 0 \ 0 \ 0 \ 2\hat{t} \\ \{\hat{x}, \widehat{xp}\} = 0 \ \pm \ 0 \ \pm \ 2\hat{p} \\ \{\hat{x}, \widehat{xpt}\} = \mp \ \pm \ \mp \ \pm \ 2\widehat{pt} \\ \{\widehat{xt}, \widehat{xp}\} = \pm \ \pm \ \pm \ \pm \ 2\widehat{pt} \\ \{\widehat{xt}, \widehat{xpt}\} = 0 \ 0 \ 0 \ 0 \ 2\hat{p} \\ \{\widehat{xp}, \widehat{xpt}\} = 0 \ + \ 0 \ - \ 2\hat{t} \end{array} \quad (107)$$

later in this section.

Write  $c_\alpha$  as a sum over canonical basis elements:

$$c_\alpha = c_1 + c_x\hat{x} + \dots, \quad (108)$$

where the  $c_\chi$  are real numbers. According to the first line in Eq. (92),  $c_\alpha$  commutes with  $\hat{t}$ . This eliminates

any term in the above that includes an odd number of spatial vectors. We can therefore restrict the form of  $c_\alpha$  to:

$$c_\alpha = c_1 + c_t \hat{t} + c_{xy} \widehat{xy} \dots \quad (109)$$

The other lines in Eq. (92) indicate that  $c_\alpha$  must become  $c_\alpha^{-1}$  when commuted with any spatial vector. The term  $c_{xy} \widehat{xy}$  becomes  $-c_{xy} \widehat{xy}$  when commuted with  $\hat{x}$ , but is unchanged when commuted with  $\hat{z}$ . Thus this term is excluded, at least for 3 or more spatial dimensions, and the same argument excludes any term with anything other than all the spatial vectors or none of them. This restricts  $c_\alpha$  to:

$$c_\alpha = c_1 + c_t \hat{t} + c_p \hat{p} + c_c \widehat{pt}, \quad (110)$$

which is equivalent to requiring that  $c_\alpha$  not distinguish spatial orientations.

From Eq. (92) we have that

$$(\widehat{xc}_\alpha)(\widehat{xc}_\alpha) = \hat{x}^2 c_\alpha^{-1} c_\alpha = \hat{x}^2 = \pm 1. \quad (111)$$

Substituting Eq. (110) and comparing like terms gives:

$$\begin{aligned} \hat{x}^2 c_1^2 + \widehat{xt}^2 c_t^2 + \widehat{xp}^2 c_p^2 + \widehat{cpt}^2 c_c^2 &= \pm 1 \\ \{\widehat{x}, \widehat{xt}\} c_1 c_t + \{\widehat{xp}, \widehat{cpt}\} c_p c_c &= 0 \\ \{\widehat{x}, \widehat{xp}\} c_1 c_p + \{\widehat{xt}, \widehat{cpt}\} c_t c_c &= 0 \\ \{\widehat{x}, \widehat{cpt}\} c_1 c_c + \{\widehat{xt}, \widehat{xp}\} c_t c_p &= 0 \end{aligned} \quad (112)$$

Using the anticommutator relations of Eq. (107), some of these four equations become  $0 = 0$ . The remainder are:

$M \bmod(4)$	then	
is 0	$+c_1^2 \pm c_t^2 - c_p^2 \mp c_c^2 = 1,$	
"	$-c_1 c_c + c_t c_p = 0,$	
is 1	$+c_1^2 \pm c_t^2 \pm c_p^2 - c_c^2 = 1,$	
"	$0c_1 c_t + c_p c_c = 0,$	
"	$+c_1 c_p + 0c_t c_c = 0,$	
"	$+c_1 c_c + c_t c_p = 0,$	(113)
is 2	$+c_1^2 \pm c_t^2 + c_p^2 \pm c_c^2 = 1,$	
"	$-c_1 c_c + c_t c_p = 0,$	
is 3	$+c_1^2 \pm c_t^2 \mp c_p^2 + c_c^2 = 1,$	
"	$0c_1 c_t - c_p c_c = 0,$	
"	$+c_1 c_p + 0c_t c_c = 0,$	
"	$+c_1 c_c + c_t c_p = 0,$	

where the number of  $\pm$  have been reduced by multiplying each line by  $\pm 1$ . The above equations are as compatible for  $-c_\alpha$  as they are for  $c_\alpha$ , but there doesn't appear to be any physical significance to negating the speed of light. So we will require the  $c_1 > 0$  and avoid the interesting but presumably unphysical situation where  $c_1 = 0$ . For cases  $M \equiv 1, 3 \bmod(4)$ , this restriction forces  $c_p = c_c = 0$ , and the equations are reduced to:

$M \bmod(4)$	then	
is 0	$c_1^2 \pm c_t^2 - c_p^2 \mp c_c^2 = 1,$	
"	$c_1 c_c = c_t c_p,$	
is 1	$c_1^2 \pm c_t^2 = 1,$	(114)
is 2	$c_1^2 \pm c_t^2 + c_p^2 \pm c_c^2 = 1,$	
"	$c_1 c_c = c_t c_p,$	
is 3	$c_1^2 \pm c_t^2 = 1,$	

Cases  $M \equiv 1, 3 \bmod(4)$ , have only one degree of freedom which we will label as  $\alpha_t$ . A general solution for these cases is

$$\begin{aligned} c_1 = \cos(\alpha_t), \quad c_t = \sin(\alpha_t) & \quad \text{if } \hat{t}^2 = -1, \\ c_1 = \cosh(\alpha_t), \quad c_t = \sinh(\alpha_t) & \quad \text{if } \hat{t}^2 = +1. \end{aligned} \quad (115)$$

This is also a solution for  $M \equiv 0, 2 \bmod(4)$  but is not general. These two forms can be combined into one by noting that

$$\exp(\beta \hat{\chi}) = \begin{cases} \cos(\beta) + \sin(\beta) \hat{\chi} & \text{if } \hat{\chi}^2 = -1, \\ \cosh(\beta) + \sinh(\beta) \hat{\chi} & \text{if } \hat{\chi}^2 = +1 \end{cases} \quad (116)$$

for  $\hat{\chi}$  any element that squares to  $\pm 1$  and  $\beta$  any real (or complex) number.

To find the general solution for  $M \equiv 0, 2 \bmod(4)$ , we follow the example of the 1, 3 cases. In either case, we need  $c_1 c_c = c_t c_p$ , so we make the substitution:

$$\begin{aligned} c_1 &= AX, & c_c &= BY, \\ c_t &= BX, & c_p &= AY. \end{aligned} \quad (117)$$

This arranges for  $c_1 c_c = c_t c_p$  automatically leaving only:

$$A^2 X^2 \pm B^2 X^2 + \kappa (A^2 Y^2 \pm B^2 Y^2) = 1, \quad (118)$$

where  $\kappa = -1$  for  $M \equiv 0$  and  $\kappa = +1$  for  $M \equiv 2$ . This can be factored to give:

$$(A^2 \pm B^2)(X^2 + \kappa Y^2) = 1, \quad (119)$$

Thus a general, two parameter solution is:

$$\begin{aligned} A &= \cos(\alpha_t) \text{ or } \cosh(\alpha_t), \\ B &= \sin(\alpha_t) \text{ or } \sinh(\alpha_t), \\ X &= \cos(\alpha_p) \text{ or } \cosh(\alpha_p), \\ Y &= \sin(\alpha_p) \text{ or } \sinh(\alpha_p), \end{aligned} \quad (120)$$

where the cos or cosh is to be taken appropriate to the signs of  $\pm$  and  $\kappa$ . Again applying the exponential formula Eq. (116), we obtain the general solution for possible values of  $c_\alpha$  for an arbitrary space-time metric as:

$$c_\alpha(\alpha_t, \alpha_p) = \begin{cases} \exp(\alpha_t \hat{t} + \alpha_p \hat{p}) & \text{if } M \text{ is even,} \\ \exp(\alpha_t \hat{t}) & \text{otherwise,} \end{cases} \quad (121)$$

where  $M$  is the number of spatial dimensions and  $\alpha_p$  and  $\alpha_t$  are arbitrary real numbers.

It should be noted that our parameterization for  $c_\alpha$  satisfies the usual relations of the exponential:

$$\begin{aligned} c_\alpha^r(\alpha_t, \alpha_p) &= c_\alpha(r\alpha_t, r\alpha_p), \\ c_\alpha^{-1}(\alpha_t, \alpha_p) &= c_\alpha(-\alpha_t, -\alpha_p). \end{aligned} \quad (122)$$

Thus the square root and inverse of  $c_\alpha$  exists.

## L. Handedness and Orientation

Up to this point this paper has used the Pauli spin operators to define Schwinger's measurement algebra for



the electron and neutrino. This was done so that we could use the Stern-Gerlach experiment for example calculations. The symmetry breaking in weak isospin depends on handedness so we need to modify our nomenclature somewhat. In addition, this will better separate the internal degrees of freedom for the particle from its external degrees of freedom, and provide a way for determining the geometric algebraic values for the spin and handedness operators.

The handedness of a particle does not depend on the direction of travel of the particle, but is an internal attribute like  $\mu_C$  and  $\mu_F$ . This eliminates orientation  $A$  from the internal attributes of the particle and is quite natural. A complete set of handed electron family lepton primitive measurements is:

$$\begin{array}{cccc} M(eR) & M(\bar{e}R) & M(eL) & M(\bar{e}L) \\ M(\nu R) & M(\bar{\nu}R) & M(\nu L) & M(\bar{\nu}L), \end{array} \quad (123)$$

where, for example,  $M(\bar{\nu}L)$  is a measurement that passes left-handed anti neutrinos traveling in any direction. The presence of  $L$  or  $R$  will distinguish these types of measurement from the states split according to spin that we used before. The internal degree of freedom that previously allowed a lepton to have two values for spin in the  $A$  direction now appears as two possibilities for handedness.

Given that we now have two measurement algebras for the electron family of leptons, Eq. (16) and Eq. (123), we should define the relationship between them. The measurements that pass only particles of a particular type ignore spin or handedness so we have:

$$\begin{array}{l} M(e) = M(eA) + M(e\bar{A}) = M(eR) + M(eL), \\ M(\bar{e}) = M(\bar{e}A) + M(\bar{e}\bar{A}) = M(\bar{e}R) + M(\bar{e}L), \\ M(\nu) = M(\nu A) + M(\nu\bar{A}) = M(\nu R) + M(\nu L), \\ M(\bar{\nu}) = M(\bar{\nu}A) + M(\bar{\nu}\bar{A}) = M(\bar{\nu}R) + M(\bar{\nu}L), \end{array} \quad (124)$$

and these are unchanged in terms of  $\mu_F$  and  $\mu_C$  as in Eq. (23). For example,  $M(e) = (1 + \mu_F)(1 + \mu_C)/4$ . We can also define the  $\mu_F$  and  $\mu_C$  operators in terms of handedness measurements as in Eq. (21), but instead of the  $\sigma_A$  operator, we will have a handedness operator. In terms of the new measurements we have:

$$\begin{array}{l} \mu_H = +M(eR) + M(\bar{e}R) - M(eL) - M(\bar{e}L) \\ \quad + M(\nu R) + M(\bar{\nu}R) - M(\nu L) - M(\bar{\nu}L), \\ \mu_F = +M(eR) + M(\bar{e}R) + M(eL) + M(\bar{e}L) \\ \quad - M(\nu R) - M(\bar{\nu}R) - M(\nu L) - M(\bar{\nu}L), \\ \mu_C = +M(eR) - M(\bar{e}R) + M(eL) - M(\bar{e}L) \\ \quad + M(\nu R) - M(\bar{\nu}R) + M(\nu L) - M(\bar{\nu}L). \end{array} \quad (125)$$

where we introduce the handedness operator  $\mu_H$ .

It is useful to define the oriented primitive measurements Eq. (16), in terms of the handed primitive measurements Eq. (123). For this purpose, we need not distinguish the particles so define:

$$\begin{array}{l} H(R) = \frac{1}{2}(1 + \mu_H), \\ H(L) = \frac{1}{2}(1 - \mu_H), \\ S(r) = \frac{1}{2}(1 + \sigma_r), \\ S(\bar{r}) = \frac{1}{2}(1 - \sigma_r) = S(-r). \end{array} \quad (126)$$

These operators split the whole state space in the sense that they pairwise add to unity:

$$H(R) + H(L) = S(r) + S(\bar{r}) = 1. \quad (127)$$

Let  $Z$  remind us of Zitterbewegung and define

$$Z(r) = H(R)S(r), \quad (128)$$

then  $Z(\bar{r}) = H(R)S(\bar{r})$  and using the annihilation relation for measurements Eq. (12), we have:

$$H(R) = Z(r)S(r) + Z(\bar{r})S(\bar{r}). \quad (129)$$

As a product in the Schwinger measurement algebra,  $Z(r)S(r)$  corresponds to two measurements. The first measures spin in the  $A$  direction and passes the particle if it comes up  $+1/2$ . The second measurement converts the spin $+1/2$  particle to a right handed particle and therefore its measurement is that of particle velocity in the  $A$  direction. This implies that

$$H(L) = Z(\bar{r})S(r) + Z(r)S(\bar{r}). \quad (130)$$

As a complete set of measurements in the Schwinger algebra,  $Z(r)$  and  $Z(\bar{r})$  should satisfy the usual axioms:

$$\begin{array}{l} Z(r)Z(r) = Z(r), \\ Z(r)Z(\bar{r}) = Z(\bar{r})Z(r) = 0, \\ Z(r) + Z(\bar{r}) = 1. \end{array} \quad (131)$$

Intuitively,  $Z(r)$  should commute with handedness and we will so assume. We can therefore write the oriented measurements in terms of the handed measurements. Collecting up all the conversions we have:

$$\begin{array}{l} H(R) = Z(r)S(r) + Z(\bar{r})S(\bar{r}), \\ H(L) = Z(\bar{r})S(r) + Z(r)S(\bar{r}), \\ S(r) = Z(r)H(R) + Z(\bar{r})H(L), \\ S(\bar{r}) = Z(\bar{r})H(R) + Z(r)H(L), \end{array} \quad (132)$$

where we have taken advantage of Eq. (131) and the assumption that  $Z(r)$  commutes.

Let  $\psi(x, y, z, t)$  be a GA valued function of the space-time manifold that is moving with speed  $v$  in the  $\hat{z}$  direction. In the section on Stern-Gerlach interference we found that the imaginary constant that appears in the interference is  $i = \sigma_x \sigma_y \sigma_z$ . In order to preserve the probability norm, this must also be the form of the interference that appears as a result of changes in path length. Therefore, a general sinusoidal solution for a plane wave moving in the  $+z$  direction is:

$$\begin{array}{l} \psi(x, y, z, t) = \cos(vt - z) \psi_0 + \sin(vt - z) \sigma_x \sigma_y \sigma_z \psi_0 \\ = \cos(vt - z) \psi_0 + \sin(vt - z) \hat{i} \psi_0, \\ = \exp(\hat{i}(vt - z)) \psi_0. \end{array} \quad (133)$$

where  $\psi_0$  is an element of the complexified Geometric Algebra and where we will from here on make the assumption

$$\hat{i} = \sigma_x \sigma_y \sigma_z. \quad (134)$$

Applying the Dirac equation to this plane wave we obtain:

$$\begin{aligned} 0 &= (\hat{t}\partial_t - \hat{x}\partial_x - \hat{y}\partial_y - \hat{z}\partial_z) \exp(\hat{i}(vt - z)) \psi_0, \\ &= (v\hat{t} + \hat{z}) \exp(\hat{i}(vt - z)) \psi_0, \\ &= (v\hat{t} + \hat{z}) (\cos(vt - z) + \sin(vt - z)\hat{i}) \psi_0. \end{aligned} \quad (135)$$

For general values of  $\psi_0$ ,  $\hat{i}$  and  $v$ , the right hand side is a function of time. In order to make this zero, we must have two separate equalities:

$$\begin{aligned} (v\hat{t} + \hat{z}) \psi_0 &= 0 \quad \text{and} \quad r_l \\ (v\hat{t} + \hat{z}) \hat{i}\psi_0 &= 0. \end{aligned} \quad (136)$$

As far as annihilating  $(v\hat{t} + \hat{z})$  on the right, a Geometric Algebra element such a  $\psi_0$  or  $\hat{i}\psi_0$  need only be described as to how it factors into  $\hat{t}$  and  $\hat{z}$  terms on the left. Accordingly, we look for real constants  $A$ ,  $B$ ,  $X$  and  $Y$  with

$$\begin{aligned} 0 &= (v\hat{t} + \hat{z})(A + X\hat{t} + Y\hat{z} + D\hat{z}\hat{t}) \\ &= (\mp vX \pm Y) + (vA \pm B)\hat{t} \\ &\quad + (A \pm vB)\hat{z} + (-vY + X)\hat{z}\hat{t}. \end{aligned} \quad (137)$$

Each of the collected terms in the above must separately be equal to zero. Thus we have that  $|v| = 1$  and the plane wave must travel at the (scalar) speed of light. Since we are looking for  $Z(r)$  rather than  $Z(\bar{r})$ , we will use  $v = 1$ . The restriction on  $A$ ,  $B$ ,  $X$  and  $Y$  is then:

$$B = \mp A, \quad \text{and} \quad Y = X, \quad (138)$$

and so both  $\psi_0$  and  $\hat{i}\psi_0$  must be of form

$$(X(\hat{t} + \hat{z}) + A(1 \mp \hat{z}\hat{t})) \iota_\chi, \quad (139)$$

where  $\iota_\chi$  is an element of the algebra that corresponds to the particle that we are propagating and is therefore a primitive idempotent. We can always factor a  $\hat{t}$  out of  $\chi_0$  on the left and use it to turn the  $A$  term in the above into  $X$  form. That is,  $(1 \mp \hat{z}\hat{t}) \hat{t} = (\hat{t} + \hat{z})$ . So we have that

$$\psi_0 = (\hat{t} + \hat{z}) \iota_\chi. \quad (140)$$

The above form will ensure that  $\psi_0$  is annihilated by  $(\hat{t} + \hat{x})$  as required. In order for  $\hat{i}\psi_0$  to also be annihilated, we must ensure that when  $\hat{i}$  is commuted around  $(\hat{t} + \hat{z})$ , it effects the  $\hat{t}$  and  $\hat{z}$  terms the same. That is, we must have:

$$(\hat{t} + \hat{x}) \hat{i} = \begin{cases} +\hat{i}(\hat{t} + \hat{z}) & \text{or} \\ -\hat{i}(\hat{t} + \hat{z}), \end{cases} \quad (141)$$

a subject we will return to after defining  $Z(r)$ .

Eq. (140) defines the action of  $Z(r)$ . That is,  $Z(r)$  should pick out the wave functions that are traveling at speed 1 in the  $\hat{r}$  direction and must be idempotent:

$$\begin{aligned} Z(r) (\hat{t} + \hat{z})\chi &= (\hat{t} + \hat{z})\chi, \\ Z(r) Z(r) &= Z(r), \end{aligned} \quad (142)$$

for  $\chi$  any element of the Geometric Algebra. These equations can be solved by a technique similar to that following Eq. (136) and the result is:

$$Z(r) = \frac{1}{2}(1 \mp \hat{r}\hat{t}). \quad (143)$$

In the standard model, the antiparticles act like particles traveling backwards in time. A plane wave traveling backwards in time in the  $+z$  direction is given by:

$$\psi(x, y, z, t) = \exp(\hat{i}(-t - z)) \phi_0. \quad (144)$$

Applying the Dirac operator as before, we obtain

$$0 = (-\hat{t} + \hat{z}) (\cos(-t - z) + \sin(-t - z)\hat{i}) \phi_0. \quad (145)$$

This is identical to Eq. (135) except for the sign of  $\hat{t}$  and  $t$ . In analogy with Eq. (140) the projection operator that picks out the antiparticles is therefore given by:

$$\frac{1}{2}(1 \pm \hat{r}\hat{t}). \quad (146)$$

The projection operator for particles is just 1 minus the above so we have:

$$\frac{1}{2}(1 + \mu_C) = \frac{1}{2}(1 \mp \hat{r}\hat{t}) = Z(r). \quad (147)$$

When we treat the antiparticles as particles traveling backwards in time, the operator that distinguishes between particles and antiparticles ends up depending on orientation. We therefore have the geometric definition of the  $\mu_C$  operator as:

$$\mu_{Cr} = \mp \hat{r}\hat{t} = \hat{r}\nu_C \quad \text{where } \nu_C = \mp \hat{t}. \quad (148)$$

In the above we have replaced  $\mu_C$  with  $\mu_{Cr}$  to designate the dependency on orientation and then factored the orientation out to leave  $\nu_C$ .

Since writing the 32 canonical basis elements listed in Eq. (26) we have made three changes. First, by eliminating the spin degrees of freedom, we obtained a handedness degree of freedom  $\mu_H$ , the Stern-Gerlach interference factor  $\hat{i}$ , and promoted the antiparticle degree of freedom  $\mu_C$  to an oriented antiparticle degree of freedom  $\mp \hat{r}\hat{t}$ . We haven't gained or lost any degrees of freedom. The new 32 elements still square to  $+1$  or  $-1$  so they still form a canonical basis set for the Geometric algebra considered as a vector space. Only the  $\mu_F$  has been left unaltered. The handed canonical basis set is:

$$\begin{array}{cccc} 1, & \mu_F, & \mu_H, & \mu_H\mu_F, \\ \hat{i}, & \hat{i}\mu_F, & \hat{i}\mu_H, & \hat{i}\mu_H\mu_F, \\ \hat{x}, & \hat{x}\mu_F, & \hat{x}\mu_H, & \hat{x}\mu_H\mu_F, \\ \hat{x}\hat{i}, & \hat{x}\hat{i}\mu_F, & \hat{x}\hat{i}\mu_H, & \hat{x}\hat{i}\mu_H\mu_F, \\ \hat{y}, & \hat{y}\mu_F, & \hat{y}\mu_H, & \hat{y}\mu_H\mu_F, \\ \hat{y}\hat{i}, & \hat{y}\hat{i}\mu_F, & \hat{y}\hat{i}\mu_H, & \hat{y}\hat{i}\mu_H\mu_F, \\ \hat{z}, & \hat{z}\mu_F, & \hat{z}\mu_H, & \hat{z}\mu_H\mu_F, \\ \hat{z}\hat{i}, & \hat{z}\hat{i}\mu_F, & \hat{z}\hat{i}\mu_H, & \hat{z}\hat{i}\mu_H\mu_F. \end{array} \quad (149)$$

The number of degrees of freedom, 32, are unchanged, however we have provided geometric values (i.e.  $\hat{x}, \hat{y}, \hat{z}$ ) for operators that were previously defined only in terms of measurements (i.e.  $\sigma_x, \sigma_y, \sigma_z$ ).

Since we have  $32 = 2^{4+1}$  degrees of freedom in the canonical basis elements listed in Eq. (149), we must have at least 4 spatial dimensions. This is one more than the usual, and is reminiscent of the Kaluza-Klein theories. We will label the extra coordinate  $s$ . The associated canonical basis vector is  $\hat{s}$  which squares as a spatial vector:

$$\hat{s}^2 = \pm 1. \tag{150}$$

The full set of canonical basis elements for the Geometric Algebra is therefore:

$$\begin{array}{cccc} 1, & \hat{x}, & \hat{y}, & \hat{z}, \\ \hat{t}, & \widehat{xt}, & \widehat{yt}, & \widehat{zt}, \\ \hat{s}, & \widehat{xs}, & \widehat{ys}, & \widehat{zs}, \\ \widehat{st}, & \widehat{xst}, & \widehat{yst}, & \widehat{zst}, \\ \widehat{xyz}, & \widehat{yz}, & \widehat{xz}, & \widehat{xy}, \\ \widehat{xyzt}, & \widehat{yzt}, & \widehat{xzt}, & \widehat{xyt}, \\ \widehat{xyzs}, & \widehat{yzs}, & \widehat{xzs}, & \widehat{xys}, \\ \widehat{xyzst}, & \widehat{yzst}, & \widehat{xzst}, & \widehat{xyst}. \end{array} \tag{151}$$

We now return to Eq. (141), the problem of arranging for  $\hat{i}$  to either commute or anticommute with  $(\hat{t} + \hat{x})$ . This implies that  $\hat{i}$  can either have  $\widehat{xt}$  or neither  $\hat{x}$  nor  $\hat{t}$ . The same argument applies to the  $\hat{y}$  and  $\hat{z}$  just as well as the  $\hat{x}$ , so we conclude that  $\hat{i}$  must either have a factor of  $\widehat{xyzt}$  or no factor of  $\hat{x}, \hat{y}, \hat{z}$  or  $\hat{t}$ . Examining Eq. (151), and recalling that  $\hat{i}$  must square to  $-1$ , we see that the only possibilities are  $1, \hat{s}, \widehat{xyzt}$  or  $\widehat{xyzst}$ . Of these,  $\widehat{xyzt}$  always squares to  $-1$ , while  $\hat{s}^2 = \pm 1$  and  $\widehat{xyzst}^2 = \mp 1$ . The three available cases for  $\vec{\sigma}$  are then:

$$\begin{array}{ccc} (- + + + +) & \text{either} & (+ - - - -) \\ \sigma_r = & \widehat{rst} & \widehat{rst} \\ \sigma_x = & \widehat{xst} & \widehat{rst} \\ \sigma_y = & \widehat{yst} & \widehat{rst} \\ \sigma_z = & \widehat{zst} & \widehat{rst} \\ \hat{i} = & \widehat{xyzst} & \widehat{rst} \end{array} \tag{152}$$

All three columns provide faithful representations of  $SU(2)$ .

however not all are compatible with the definition of  $\mu_F$  and  $\mu_H$ .

FIG. 3: Table of standard model fermion quantum numbers.

	$t_3$	$t_0$	$Q$	$Q'\sqrt{3}/2$
$e_R$	0	-1	-1	1/2
$e_L$	-1/2	-1/2	-1	-1/2
$\nu_L$	1/2	-1/2	0	1
$\nu_R$	0	0	0	0

The satisfaction of the Dirac equation led to an annihilation relation between  $\nabla$  and  $\psi$  that required the result:

$$(\hat{t} + \hat{z})(\hat{t} + \hat{z}) = 0. \quad (153)$$

Thus  $(\hat{t} + \hat{z})$  is nilpotent and  $\psi_0$  is the product of a nilpotent part on the left and an idempotent part (i.e.  $\iota_\chi$ ) on the right. In the usual spinor representation of QFT, the idempotents are ignored and only the nilpotent part of the algebra is used. This could be the geometric origin of the Grassmann algebra used in the standard model.

## APPENDIX A: THE LOUNESTO GROUP

Other than  $\hat{1}$ , the remaining 31 canonical basis elements have eigenvalues of  $-1$  as well as  $+1$ . The multiplicity of each is sixteen, and the eigenvectors are quick to write down. For example, the eigenvectors of  $\widehat{xyt}$  with

eigenvalue  $-1$  are given by:

$$\widehat{xyt}(1 - \widehat{xyt})\eta = -(1 - \widehat{xyt})\eta, \quad (A1)$$

where  $\eta$  is any Clifford algebra constant that doesn't annihilate  $(1 - \widehat{xyt})$ . Sixteen values for  $\eta$  that give linearly independent eigenvectors include any nonzero term in the product  $(1 + \hat{x} + \hat{y} + \hat{t})(1 + \hat{z})(1 + \hat{s})$ .

Clifford algebra elements of the form  $(1 \pm e)/2$  are idempotent, if  $e$  is any element that gives 1 when squared. If  $e_1, e_2$  and  $e_3$  mutually commute and square to 1, then

$$\iota = (1 \pm e_1)(1 \pm e_2)(1 \pm e_3)/8, \quad (A2)$$

where the  $\pm$  are to be taken independently, are eight idempotents. If  $e_1, e_2$ , and  $e_3$  are canonical basis elements and generate a group of size 8 under multiplication, then these eight idempotents are distinct, and in the context of the Clifford algebra used here, are mutually annihilating idempotents.[5] If the set is complete in that there are no other nontrivial roots of unity that commute with  $e_1, e_2$  and  $e_3$ , then these eight idempotents are primitive idempotents.[5] An example of a set of  $e$  that satisfy these requirements is  $e_1 = \hat{zt}$ ,  $e_2 = \hat{ixy}$ ,  $e_3 = \hat{s}$ .

Out of respect for the contributions of Pertti Lounesto to the mathematical understanding of spinors, we will refer to a set of  $e_n$  that satisfy these requirements:

$$\begin{aligned} e_n e_m &= e_m e_n, \\ e_n &\in \text{Canonical basis}, \\ e_n^2 &= 1, \\ \{e_n\}_{n=1}^3 &\text{ generates a group of order 8,} \end{aligned} \quad (A3)$$

as a set of ‘‘Lounesto group generators’’. The group of size 8 will then be the ‘‘Lounesto group’’.

Since the Lounesto group generators commute, so do their various products. Therefore Eq. (A2) gives primitive idempotents that possess good quantum numbers with respect to all eight Lounesto group elements. Seven of the Lounesto group elements are nontrivial, the trivial one is 1.

The choice of generators among the Lounesto group is somewhat arbitrary. So long as we pick three group elements that are distinct and nontrivial, and we avoid picking a set that multiplies to unity, our three elements will generate the other five. Translated back into physics, this means that we need only keep track of three quantum numbers for each of the primitive idempotents. The rest are related by multiplication.

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