Nonlinear Waves On the Geometric Algebra

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Soliton solutions of nonlinear wave equations are postulated as a model for the elementary chiral fermions. The Dirac equation is derived from the requirement that the solitons be represented in a non interacting form, and that the nonlinear wave equation be of a reasonable type in the Geometric Algebra.

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This is the fifth paper in a series by the author describing a new foundation for quantum mechanics based on the Proper Time Geometry (PTG). For an introduction to the PTG and a brief discussion of how clasical relativistic mechanics works in the geometry, see [1]. For an introduction to the Geometric Algebra, see [2]. For a geometric description of the internal symmetries of the fermions, see [2]. For an explanation of how the symmetries of charge conjugation and parity complementation come to be violated, see [3]. For a brief description of how the PTG explains the issue of de Broglie's matter waves having a phase velocity exceeding c, see [4]. These earlier papers were mostly dedicated to either classical mechanics or the internal symmetries of fermions. This paper is devoted to the question of soliton plane waves in the context of the PTG. For work by other authors using similar modifications of the geometry of special relativity see [5], [6], [7], [8].

The PTG is an alternative geometry for Special Relativity (SR) that shares the same local metric equation as the LMG, but a different interpretation of the coordinates, and a different global topology.[1] A previous paper by the author, [2], gave a geometric interpretation of the fermions in terms of the Geometric Algebra (GA) defined on the PTG. A GA is a Clifford algebra generated by a vector basis given by the tangent space of the underlying manifold, with a signature chosen to match that of the natural metric for the manifold. For an introduction to how GAs are used in modeling classical mechanics, see [9].

Given the history of quantum mechanics, it is natural for physicists to use sine waves, that is, waves of the form $\exp(i(k\cdot r-\omega t))$, to model matter waves. That is, one assumes that the free particle momentum eigenstates correspond to sine waves, and one then expands an arbitrary problem in terms of sine waves. Instead, this paper will suggest that the free particle momentum eigenstates are square waves. That is, they are periodic waves which consist of constant values separated by discontinuities. This radical departure from standard physics requires some justification.

The most important reason for considering square waves instead of sine waves is the formalism of quantum field theory (QFT). In QFT, calculations are particularly simple when done in the momementum representation, that is, when done with momentum eigenstates. This suggests that these momentum eigenstates should have a particularly simple model. The problem is that the annihilation and creation operators act to replace one momentum eigenstate with another. This is apparently a discontinuous operation. What's worse, in the position representation the replacement happens simultaneously over all space.

In the context of standard QFT, the replacement of one sine wave by another simultaneously over all space makes some sort of sense, in that the individual particles are assumed each to have use of their own copy of the Dirac equation. Before the particle exists, there is no wave, nor after it is gone. On the other hand, a theme of the author's series of papers is an effort at unifying the wave equations for the various particles into a single equation. In the context of a unified wave equation, the transformation of sine waves into one another (for example, with different wave vectors) is a discontinuous process that involves instantaneously changing the unified wave function over all space.

Square waves, by contrast to sine waves, can be added together to produce non periodic sums. This ability is reminiscent of the Feynman diagrams where a fermion and antifermion combine to produce a boson. As far as discontinuities, the process of combining two square waves so as to eliminate their periodicity is a discontinuous process, but as with the propagation of square waves, the discontinuity is restricted to a set of measure zero. This will be the topic of the next paper in this series.

No real world particles have exact momenta and so the momementum eigenstates are an idealization only. For a practical theory, we therefore need not worry that we are requiring the universe to possess even the mild discontinuities present in these ideal square waves. What we are doing here is accepting a very mild form of discontinuity in our waves in return for eliminating a very strong form of discontinuity in how our waves interact with each other.

Another reason to consider square waves is that they can be used to solve wave equations for nonlinear wave equations. While the wave equations of standard quantum mechanics are linear, this linearity is clearly violated when they are given a probabilistic interpretation.

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This series of papers will eventually address the measurement problem, that is, the problem of uniting the wave and particle identities of particles, and for this, a theory compatible with nonlinear wave theory will be at an advantage.

"The Geometry of Fermions"[2] proposed that the fermions can be associated with elements of the GA that satisfy the nonlinear equation:

$$e_{nlm}^2 = e_{nlm}, \tag{1}$$

where n, l and m are indices indicating the type of fermion. Now consider a polynomial over the GA:

$$F(\chi) = \sum_j \alpha_j \chi^j, \tag{2}$$

with α_j and χ taken over the GA. Note that the polynomial reduces to linear form when χ satisfies $\chi^2 = \chi$. Thus the structure of the fermions themselves suggests that we should consider nonlinear equations that involve polynomials such as that of Eq. (2) as possible wave equations. It is also interesting to note that Eq. (1) originally comes from Schwinger's "Measurement Algebra". Thus nonlinearity is related to measurement.

I. SCALAR WAVES ON $(x;t)$.

It was earlier shown[2, App.] that the Dirac equation can be "derived" in that it is the simplest linear wave equation that can be written down in the PTG using its GA. The Klein Gordon wave equation can be obtained by squaring the Dirac wave equation, so it too can be derived this way. But it should be clear that nature is not at all linear. The Pauli exclusion principle makes this very apparent, as does the probability interpretation of wave functions. The best argument for the use of linear equations is that they are easier to solve than more general equations.[10]

As an example of a nonlinear wave equation, consider the scalar equation:

$$\partial_t \psi(x,t) = \partial_x(\psi^2) = 2\psi\partial_x\psi. \tag{3}$$

Before we consider the complication of a wave equation over the PTG that takes values over a geometric algebra instead of the reals, we will examine the soliton solutions to this simple equation.

Let us look for solutions to Eq. (3) that move with no change in shape, at constant speed v in the $+x$ direction. We will call these wave equation solutions "soliton" waves. By the assumption of constant speed with no change in shape, ψ satisfies, in addition to Eq. (3), the requirement:

$$\psi(x,t) = \psi(x - vt, 0) = \psi_0(x/v - t), \tag{4}$$

where ψ_0 gives the initial condition for ψ. Taking derivatives with respect to x and t gives:

$$\begin{aligned} \partial_x\psi(x,t) &= \dot{\psi}_0(x/v - t)/v, \\ \partial_t\psi(x,t) &= -\dot{\psi}_0(x/v - t). \end{aligned} \tag{5}$$

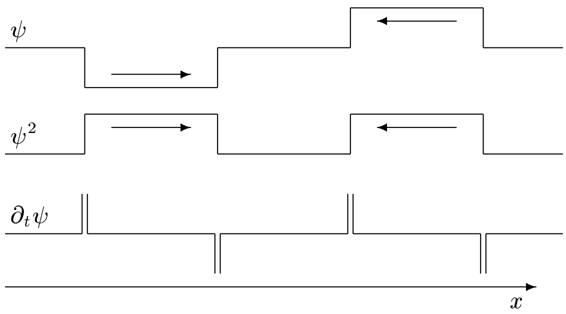


FIG. 1: Soliton wave solutions for Eq. (3). Two waves are shown, prior to a collision.

Substituting these into Eq. (3), and setting $t = 0$, gives:

$$-\dot{\psi}_0(x) = 2\psi_0\dot{\psi}_0/v. \tag{6}$$

This equation will be satisfied when either $\dot{\psi}_0 = 0$, or $\psi_0 = -v/2$.

It is evident that we will have to consider discontinuous waves in order to realize a nontrivial solution. Accordingly, assume that ψ_0 has an isolated discontinuity at $x = 0$ and nowhere else. Thus ψ_0 will be a step function with values $\psi_0(x) = \psi_-$ or ψ_+ according as x is less than or greater than zero. This gives ψ as:

$$\begin{aligned} \psi(x,t) &= \psi_- \quad x < vt, \\ &= \psi_+ \quad x > vt \end{aligned} \tag{7}$$

This satisfies Eq. (3) except, perhaps, at $x = vt$. At that point, the derivatives with respect to x and t, will be delta functions:

$$\begin{aligned} \partial_t\psi(x,t) &= (\psi_- - \psi_+)\delta(x/v - t), \\ \partial_x\psi(x,t)^2 &= (\psi_+^2 - \psi_-^2)\delta(x - vt), \\ &= (\psi_+^2 - \psi_-^2)\delta(x/v - t)/v. \end{aligned} \tag{8}$$

Equating the two sides gives:

$$\begin{aligned} (\psi_- - \psi_+) &= (\psi_+ + \psi_-)(\psi_+ - \psi_-)/v, \text{ so} \\ -v &= \psi_+ + \psi_-. \end{aligned} \tag{9}$$

We would like to be able to add soliton solutions, so we will choose a vacuum of 0. Thus the solitons will consist of square pulses that travel with velocity v, and have heights of $-v$. To get solitons that travel in the opposite direction, we need only invert the pulses. See Fig. (1) for an illustration of positive and negative pulses. Note that the two solitons illustrated will collide and annihilate each other.

The solitons derived for Eq. (3) are unrealistic, as far as using them to model the elementary chiral fermions, for several reasons. First, the chiral fermions travel at close to c, while these solitons have arbitary speeds, including both faster and slower than light. Second, their ability to concentrate in x is a violation of the Heisenberg uncertainty principle. Third, they have nowhere near enough degrees of freedom to model the particle zoo. Fourth, they violate Lorentz symmetry. Fifth, they're only in one dimension. Using the PTG and promoting the scalars to geometric algebra elements solves these problems.

II. SCALAR WAVES ON $(x, s; t)$.

Our first modification will be to change the base space for Eq. (3) by adding the s dimension. Let $v = (v_x, v_s)$ be the direction of propagation of a soliton for a scalar wave u defined on this geometry by the following nonlinear wave equation:

$$\partial_t \phi(x, s; t) = (v_x \partial_x + v_s \partial_s)(\phi^2). \tag{10}$$

Later in this paper, when we generalize to the machinery of the geometric algebra, we will remove the dependence on v. For now, note that the gradient operator, when applied to a plane wave travelling in the v direction, will give a result whose magnitude is proportional to $v_x \partial_x + v_s \partial_s$ applied to the same plane wave. The problem with simply using the ∇ operator is that when applied to a scalar wave, it gives back a vector. In the context of the geometric algebra, which allows the mixing of scalars and vectors, this is not a problem.

In analogy with Eq. (4), we will look for solitons of the form:

$$\phi(x, s; t) = \phi_0(s/v_s + x/v_x - t), \tag{11}$$

where v_s and v_x are nonzero phase velocities. Taking derivatives with respect to x, s, and t gives:

$$\begin{aligned} v_x \partial_x \phi(x, s; t) &= \dot{\phi}_0, \\ v_s \partial_s \phi(x, s; t) &= \dot{\phi}_0, \\ \partial_t \phi(x, s; t) &= -\dot{\phi}_0. \end{aligned} \tag{12}$$

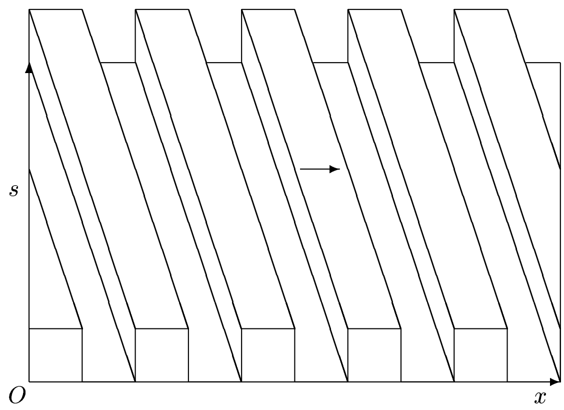
Substituting these into Eq. (10), and setting $x = s = 0$, gives:

$$-\dot{\phi}_0 = 4\phi_0 \dot{\phi}_0. \tag{13}$$

This equation will be satisfied when either $\dot{\phi}_0 = 0$ or $\phi_0 = -1/4$. As before, it is evident that we will have to consider discontinuous waves in order to achieve a nontrivial solution. But in addition, we need to keep ϕ single valued when the s coordinate changes by $2m\pi R_s$. Therefore ϕ must be periodic in x with wavelength $\lambda_x = 2\pi R_s v_x / v_s$, and periodic in time with period $\tau = 2\pi R_s / v_s$. The phase velocity in the x direction is, as advertised, $\lambda_x / \tau = v_x$. The resulting wave is illustrated in Fig. (2).

The addition of the hidden dimension has had the effect of forcing the soliton solutions to lose their ability to be concentrated in the x dimension. This can be thought of as an analog to the Heisenberg uncertainty principle.

FIG. 2: A soliton wave solution $-\phi$ for Eq. (10).



That is, when we require that a wave solution to Eq. (10) have a precise velocity, we lose the ability to simultaneously restrict the wave's spatial extent. With these results for scalar wave functions, we can now bring in the machinery of the geometric algebra.

III. THE DIRAC EQUATION.

Suppose that Ψ satisfies a nonlinear wave equation, and we wish to associate some of its soliton solutions with the elementary chiral fermions. As with previous work[2], we would like a single wave equation to be able to support the propagation of all the elementary particles. Before considering the generalization of Eq. (10), we will consider the requirement that our wave equation support the usual rotational symmetry of 3−space. That is, any candidate wave equation will have to allow non interacting solitons to propagate in all directions without interference.

To propagate without interference means that we require that we can, if we ignore interactions, decompose a wave solution into a sum over particles. Accordingly, let Ψ be composed of n non interacting soliton plane wave solutions, each with velocity vector v_n with $|v_n| = c = 1$, and wave function ψ_n so that:

$$\Psi(x, y, z, s; t) = \sum_n \psi_n(x, y, z, s; t), \tag{14}$$

The requirement that each ψ_n move unchanged with velocity v_n is equivalent to:

$$\partial_t \psi_n = -v_n \cdot \nabla \psi_n = -\nabla \cdot v_n \psi_n, \tag{15}$$

where ∇ has been dotted with v_n since that is the only direction in which ψ_n has variation.

Since v_n is an element of the Clifford algebra, it can be used as an operator. Consider its action on ψ_n (i.e. the operation of multiplication on the left by v_n). The fact that v_n squares to 1 implies that it has eigenvalues of ± 1.

It's natural to suppose that each ψ_n can be assumed to be an eigenvector of v_n.

For a wide class of nonlinear wave equations, we will have that the possible soliton waves can be separated into waves that have specific eigenvalues with respect to v_n. Consider the projection operators into these two classes:

$$P^{\pm} = 0.5(1 \pm v_n). \tag{16}$$

These operators commute with ∂_t and $v_n \cdot \nabla$, so any wave equation written only with ∂_t, ∇ and Ψ, can be separated into two wave equations, according to the eigenvalues of its solitons in the v_n direction. For example, suppose the wave equation is:

$$\partial_t \Psi = F(\nabla, \Psi), \tag{17}$$

where F can be written in a Taylor series over the reals. The projection operators, $P^{\pm}$, will project out the components of the wave Ψ with the appropriate eigenvalues, however, in order to divide the wave equation into two wave equations, one for the $+1$ eigenvalued waves, the other for the -1 eigenvalued waves, we must wait until the system evolves to a condition where these waves are no longer interacting. That is, we must choose a time, t_0 where $P^+\Psi$ and $P^-\Psi$ are nonzero only in distinct regions. At such a time, we will have that $P^{\pm}\Psi^n = \pm\Psi^n$, and we can then say that the waves have been separated into the projected classes. But even if no such time ever arises, we can still assume that the waves can be so divided, in principle.

It is therefore natural to suppose that each ψ_n is an eigenvector of v_n with eigenvalue ± 1. If so, the requirement that the noninteracting solitons be allowed to propagate independently gives us the following simple equation:

$$\partial_t \Psi^{\pm} = \pm \nabla \Psi^{\pm}, \tag{18}$$

where $\Psi^{\pm}$ are the waves with eigenvalues of ± 1. It was earlier shown[2] that the $+$ choice in the above equation gives a multiple representation of the Dirac equation. The $-$ choice is the same thing, but with time reversed.

Thus the soliton solutions of a wide variety of nonlinear wave equations can be written, in noninteracting form, as Dirac propagated waves. This suggests that Feynman diagrams can be interpreted as the result of a linearization of an underlying nonlinear wave equation. It also suggests that we examine the methodology of Feynman diagrams for hints as to the nature of the underlying nonlinear wave equation. This will be the subject of a later paper.

IV. CLIFFORD WAVES ON $(x, y, z, s; t)$.

We can now generalize the results of the previous sections to the general problem of GA valued waves defined over the PTG:

$$\partial_t \Psi(x, y, z, s; t) = \nabla(\Psi^2). \tag{19}$$

As in previous papers, we tacitly assume a notational vector $\hat{t}$ to signify the difference between position and momentum, but for the purposes of this paper, we need not explicitly take this into consideration.

Suppose that the direction of motion is given by a unit vector $\hat{k}$. We wish to find a soliton plane wave $\psi_{\pm k}$ with $\hat{k}\psi_{\pm k} = \pm\psi_{\pm k}$. Accordingly, try

$$\psi_{\pm k} = 0.5(1 \pm \hat{k})(\lambda_0 + \lambda_1)u(k \cdot r - \omega t), \tag{20}$$

where λ_0 and λ_1 are geometric algebra constants that have no $\hat{k}$ dependence (when $\hat{k}$ is considered as a generator of the geometric algebra),and are even and odd, respectively, and u is a scalar function giving the spatial dependence. In this form, $\psi_{\pm k}$ satisfies $\hat{k}\psi_{\pm k} = \pm\psi_{\pm k}$, and the division into odd and even parts eases calculations. Commuting the $(1 \pm \hat{k})$ factor around the leftmost $\lambda_0 + \lambda_1$ cancels out the term with λ_1 leaving:

$$\psi^2_{\pm k} = 0.5(1 \pm \hat{k})\lambda_0(\lambda_0 + \lambda_1)u^2(k \cdot r - \omega t). \tag{21}$$

As an aside directed to the problem of more general polynomial wave equations, note that:

$$\psi^j_{\pm k} = 0.5(1 \pm \hat{k})\lambda_0^{j-1}(\lambda_0 + \lambda_1)u^j(k \cdot r - \omega t). \tag{22}$$

Setting $\lambda_0 = 1$, while leaving λ_1 arbitrary reduces the wave equation Eq. (19), from an equation over the geometric algebra into a simple scalar differential equation:

$$\partial_t u = \partial_k(u^2) = 2u\partial_k u, \tag{23}$$

which we have already solved in the earlier sections of the paper. More generally, let $\lambda_0^2 = \lambda_0$, that is, let λ_0 be an idempotent that is even with respect to k. Then any λ_1 that is odd will give a solution of the form $\lambda_0 + \lambda_0\lambda_1$, and this substitution will also work with Eq. (22). These substitutions reduce the GA valued equation down to a scalar equation of the form already solved. Therefore, the soliton is a square wave over the GA.

At this point, we have solved most of the problems that afflicted the scalar version of solitons as a model for the elementary chiral fermions. We can choose a nonlinear wave equation that supports speeds of approximately c as long as k_s is relatively small. The presence of the Dirac equation, along with the metric that the PTG shares with Minkowski space, shows that the waves will be Lorentz invariant. The use of the geometric algebra provides sufficient degrees of freedom to model the elementary fermions. And the Heisenberg uncertainty principle cannot be violated by these waves.

We have previously shown how an ideal structure for the elementary fermions may be written in the geometric algebra on the PTG[2]. The ideals included there can also be used to separate nonlinear wave equations of the form Eq. (17). That is, any solution to such a wave equation can be multiplied on the right by the ideal corresponding to an elementary fermion. Since, in a nonlinear theory, the particles do interact, this procedure only makes sense in regions of space-time where the particles are not interacting.

A previous paper,[3] showed how the violation of parity and charge conjugation can be attributed to a psuedo scalar component to the speed of light. This was done by deriving the Dirac equation from the Klein-Gordon equation in the context of a geometric algebra defined on the PTG. But the paper also showed that the class of nonlinear wave equations of the form Eq. (17) were compatible with the transformations used to put the generalized Dirac equation into psuedoscalar form. Thus we have shown how it is possible to define a nonlinear wave equation that supports these odd features of particle physics.

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[10] The first places to look for your missing keys are the ones that are easily lit. If the only tool one has is a hammer, one should assume that all problems are nails. Etc.