The Geometric Speed of Light

Carl Brannen*
(Dated: November 7, 2004)

A derivation of the Dirac equation, from the Klein Gordon equation is performed using the tools of the Geometric (Clifford) Algebra. The electroweak violation of C, P, and T symmetry and the Weinberg angle, θ_W, are shown to result from a generalization of the scalar speed of light to a mixed scalar psuedoscalar.

PACS numbers:

This paper uses the Geometric Algebra (GA) on the Proper Time Geometry [1] to repeat Dirac's derivation of the Dirac equation from the Klein-Gordon equation. A GA is a type of Clifford Algebra. These algebras are characterized by the curious feature that they allow addition of scalars to vectors, psuedoscalars, etc. While this power might seem to have little use in physics, in fact the equations of physics are simplified by the use of a GA.[2] A hint of the possible uses for such an algebra for modeling the weak force is indicated by the presence of mixed scalar and psuedoscalar terms in the phenomenology of the weak interactions. [3, §20.2] Accordingly, this paper rederives the Dirac equation from the Klein Gordon equation with the speed of light, c, generalized to a GA constant c_α.

Dirac derived the Dirac equation by looking for a linear version of the Klein-Gordon equation:

$$(\partial_t)^2\psi = (\nabla^2 + m^2)\psi. \tag{1}$$

His effort was wildly successful and is repeated in some modern textbooks. [4][5] Other common texts do not show his derivation, but at least derive the fact that the Dirac equation implies the Klein-Gordon equation. [6] This paper will repeat Dirac's derivation, but in the context of a GA, which is a Clifford Algebra associated with a manifold. Given a manifold of dimension n, its GA will have dimension, as a linear vector space over the reals, of 2^n. The derivation here holds for other GAs, but in order to interpret the results, we will rely on the GA of the Proper Time Geometry (PTG).

The PTG is an alternative geometry to that of Minkowski, which nevertheless supports the results of special relativity.[1] Similar geometries, in that they reinterpret the Minkowski metric in a similar way[7], or consider a null subspace of space with an extra dimension[8] are described by other authors. The primary differences between these and the PTG, are that the PTG explicitly assumes that the extra dimension is physical, and has a radius, R_s that could, at least in principle, be measured in standard length units, and therefore can have integration and differentiation performed upon it, and the PTG treats time as a parameter rather than an element of the geometry.

Since the Klein Gordon equation, is purely scalar in each of its terms, it can be applied to more complicated algebras by simply reinterpreting ψ to be a function taking values from an algebra more complicated than the reals or complexes. In the context of the PTG, mass can be provided through various techniques involving resummation, renormalization, and or the momentum in the hidden dimension. These will be discussed in a later paper. For this paper we will restrict our attention to the massless Klein Gordon equation.

The PTG, as a manifold, has dimension 4, therefore the associated GA has dimension $2^4 = 16$. For this paper, the coordinates will be chosen as x, y, z, and s, where s is a cyclic coordinate with a radius of R_s. Time is not explicitly included in the geometry, but is instead interpreted as a parameter. Functions defined on this geometry therefore possess 16 degrees of freedom and are defined on a total space (including time) with a geometry of $\mathbf{R}^3 \times \mathbf{S}^1 \times \mathbf{R}$ or corresponding coordinates $(x, y, z) \times (s) \times (t)$.

In order to convert PTG wave functions to wave functions defined on the usual space-time, one takes a Fourier series to eliminate the dependence on the hidden dimension s. The elementary bosons will correspond to the 0th order term in the series, the electron family of fermions will correspond to the 1st order term, the muon family to the 2nd order term, etc. This allows a single wave equation in the PTG to possess sufficient degrees of freedom to represent all the elementary particles.

It is standard practice in modern geometry to use units where $c = 1$. This makes less sense in the PTG, where time is considered to be distinct from the geometry of space. In addition to the PTG, there are several interesting modern theories of space-time that assume a preferred frame of refernce. A good example of a modern theory that includes a preferred frame of reference, and also makes more explicit use of proper time, is the "Absolute Euclidian Space-Time" (AEST) theory of Montanus, [7]. For these theories, while there is no reason to assume that space and time are interchangeable, the wave equations are simpler when written with $c = 1$, so this is still the practice.

In the PTG theory, the elementary particles are associated with deformations of the space-time manifold, with the deformations capable of being modeled by the GA.

*Electronic address: carl@brannenworks.com; URL: http://www.brannenworks.com

The author's previous paper[9] shows that if one assumes the existence of a subquark particle called the "binon", it is possible to not only derive the $\mathcal{SU}(3)\times\mathcal{SU}(2)\times\mathcal{U}(1)$ symmetry of the elementary particles, but it is possible to derive the geometric form of the operators corresponding to charge conjugation C, parity complementation, P and time reversal T, as well as those corresponding to spin, S_x, S_y, S_z, electric charge Q_e, weak charge, Q_w, weak hypercharge, τ_0, and weak isospin, τ_1, τ_2, and τ_3. In addition, the algebraic ideals corresponding to the binons were derived. This paper will use these results.

I. DERIVATION OF THE DIRAC EQUATION

We begin with the massless Klein Gordon wave equation as defined as a GA valued function defined on the PTG plus time:

$$\partial_t^2\Psi(x,y,z,s;t) = c^2\nabla^2\Psi. \qquad (2)$$

The above is a set of 16, 2nd order differential equations. As such, its general solution has dimension 32. That is, to specify a solution, we must supply 32 initial values.

In order to convert Eq. (2) into a pair of 1st order equations, we can break the equation into two equations with the following assignments:

$$\begin{aligned}\partial_t\Psi &= \Psi',\\ \partial_t\Psi' &= c^2\nabla^2\Psi,\end{aligned} \qquad (3)$$

but this still leaves us with a 2nd order operator in the second equation. A better way to break the equation up is to postulate a pair of fully linear equations:

$$\begin{aligned}\partial_t\Psi_A &= c\nabla\Psi_B,\\ \partial_t\Psi_B &= c\nabla\Psi_A,\end{aligned} \qquad (4)$$

The above two equations are symmetric, so we really don't have a preference for which one should be associated with the original Ψ of Eq. (2), hence the symmetric labels A and B. Counting degrees of freedom, since we now have two waves, Ψ_A and Ψ_B, we still have 32 degrees of freedom in the two waves.

Note that when Dirac took the square root of the Klein Gordon equation, he was simultaneously changing the algebra on which it was defined. That is, he began with a Klein Gordon equation which used scalar real valued wave functions and he ended up with the Dirac equation that uses $4-$vector complex valued wave functions. In that context, it made sense to equate ψ_A and ψ_B in Eq. (4). In the present context, with more mathematical machinery available, we will retain the two halves of the equation as distinct parts of the total wave.

This paper is more concerned with symmetries of the wave equation Eq. (2) as a set of differential equations than the symmetries of the GA alone. So in order to simplify our notation, as well as to follow the notation of our previous paper, [9] we will add to the basis set for the GA a notational vector $\hat{t}$. This vector will be used to distinguish between Ψ_A and Ψ_B.

As an extension similar to complexification, the properties of $\hat{t}$ are that it squares to one, and commutes with all elements of the GA. With these definitions, the wave equation Eq. (4) can be rewritten as:

$$\begin{aligned}\hat{t}\partial_t(\Psi_A) &= c\nabla(\hat{t}\Psi_B),\\ \hat{t}\partial_t(\hat{t}\Psi_B) &= c\nabla(\Psi_A).\end{aligned} \qquad (5)$$

Writing $\Psi = \Psi_A + \hat{t}\Psi_B$, this puts Eq. (3) into the simple form:

$$\hat{t}\partial_t\Psi = c\nabla\Psi. \qquad (6)$$

In the above, Ψ now represents a wave with twice the degrees of freedom as the GA from which it takes values, with the extra degree of freedom defined by multiplication by $\hat{t}$. This derivation of the "Dirac" equation is distinct from the usual in that we are not throwing away half the degrees of freedom of the associated Klein Gordon equation. Our need to do this is due to the fact that we are making a geometric interpretation of each of the degrees of freedom.

Redefining the speed of light c to a GA constant c_α, and multiplying by $\hat{t}$ gives the generalization of the Dirac equation as:

$$\partial_t\Psi = c_\alpha\nabla(\hat{t}\Psi). \qquad (7)$$

The requirement that the above equation square to the Klein Gordon equation, and therefore a restriction on values of c_α is:

$$c^2\nabla^2 = c_\alpha\nabla c_\alpha\nabla. \qquad (8)$$

The above, as an operator equation, has to be true when applied to any function, so we can cancel the right most ∇ on each side:

$$c^2\nabla = c_\alpha\nabla c_\alpha. \qquad (9)$$

Of the basis elements of the GA, only the scalar terms $\hat{1}$ and $\hat{t}$ commute with ∇, and only the psuedoscalar terms $\widehat{xyzs}$ and $\widehat{xyzst}$ anticommute. The remaining basis elements neither commute nor anticommute with ∇ and can be eliminated if one considers the operator equation applied to simple functions like $f = z$. Accordingly, c_α is restricted to be of the form:

$$c_\alpha = c_1\hat{1} + c_t\hat{t} + c_p\widehat{xyzs} + c_c\widehat{xyzst}, \qquad (10)$$

where c_1, c_t, c_p, and c_c are real constants. Substituting this into Eq. (8), taking into account the commutation and anticommutation rules, multiplying out and equating the nonzero terms gives:

$$\begin{aligned}c^2 &= c_1^2 + c_t^2 - c_p^2 - c_c^2,\\ 0 &= 2c_1c_t - 2c_pc_c.\end{aligned} \qquad (11)$$

This consists of two equations in four unknowns, so we expect a solution set with two independent parameters. A convenient solution parameterized by two real valued numbers, α_p and α_c, is as follows:

$$\begin{aligned} c_1 &= c(\cosh(\alpha_p+\alpha_c)+\cosh(\alpha_p-\alpha_c))/2 \\ c_t &= c(\cosh(\alpha_p+\alpha_c)-\cosh(\alpha_p-\alpha_c))/2 \\ c_p &= c(\sinh(\alpha_p+\alpha_c)+\sinh(\alpha_p-\alpha_c))/2 \\ c_c &= c(\sinh(\alpha_p+\alpha_c)-\sinh(\alpha_p-\alpha_c))/2. \end{aligned} \quad (12)$$

In addition to this solution, as an obvious symmetry one can negate c_α. We will ignore such solutions, which correspond to a reversal in time. The usual Dirac equation, with scalar speed of light, corresponds to $\alpha_p = \alpha_c = 0$.

Now that we have a generalization to the speed of light, we will again set $c = 1$, which simplifies several interesting properties of c_α. The first is that powers of c_α are particularly easy to compute:

$$(c_\alpha(\alpha_p, \alpha_c))^m = c_\alpha(m\alpha_p, m\alpha_c), \quad (13)$$

for m any real number. Thus c_α has an inverse. Another useful property is the commutation relation between c_α and ∇:

$$c_\alpha \nabla = \nabla (c_\alpha)^{-1}. \quad (14)$$

The above is a restatement of Eq. (9). More generally, c_α commutes with even elements of the GA, and "inverse commutes" with the odd:

$$c_\alpha(\Phi_e + \Phi_o) = \Phi_e c_\alpha + \Phi_o c_\alpha^{-1}. \quad (15)$$

These relationships make calculations with c_α easy.

The Ψ of Eq. (7) is defined on the hidden dimension s. The Dirac wave function, of course, has no such dependency. To convert Eq. (7) into Dirac form requires that we take a Fourier series over the hidden dimension as is explained more fully in the author's previous paper.[9, Appendix] This breaks the equation into the fermion families:

$$\hat{t}\,\partial_t \Psi_n = (c_\alpha \nabla_3 + n/R_s\; i\hat{s}\,)\Psi_n, \quad (16)$$

where n indicates the Fourier series term, ∇_3 is the $3-$dimensional GA differential operator, R_s is the hidden dimension radius, and $n/R_s = m_n$ is an effective mass. Multiply on the left by $(c_\alpha(\alpha_p, \alpha_c))^{-0.5} = c_\alpha(-0.5\alpha_p, -0.5\alpha_c)$ to get:

$$c_\alpha^{-0.5}\hat{t}\partial_t \Psi_n = (\nabla_3 c_\alpha^{-0.5} + n/R_s\; i c_\alpha^{-0.5}\hat{s}\,)\Psi_n, \quad (17)$$

where ∇_3 has been commuted with $c_\alpha^{+0.5}$ by use of Eq. (14). Since $c_\alpha^{-0.5}$ everywhere multiplies Ψ_n on the left, and since this is just an invertible GA constant, we can take $\Psi'_n = c_\alpha^{-0.5}\Psi_n$, and multiply on the left by $\hat{s}$ to get:

$$\widehat{st}\;\partial_t \Psi'_n = (\hat{s}\nabla_3 + m_n i)\Psi'_n, \quad (18)$$

That this is equivalent to the Dirac equation can be seen by expanding $\nabla_3 = \hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z$ and using the following equivalencies:

$$\begin{aligned} \gamma^0 \equiv \widehat{st}, \quad \gamma^2 \equiv \widehat{ys}, \\ \gamma^1 \equiv \widehat{xs}, \quad \gamma^3 \equiv \widehat{zs}, \end{aligned} \quad (19)$$

and verifying that the Dirac equation's anticommutation relation is satisfied:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \quad (20)$$

Therefore, Eq. (16) is a representation of the Dirac equation. Since Ψ_n has more degrees of freedom than the ψ of a Dirac equation, it is clear that the equation is a multiple representation of the Dirac equation.

We have shown two facts. The first is that our generalization of the derivation of the Dirac equation does, in fact, produce a representation of the Dirac equation. But we have also shown what the relation is between different representations of the Dirac equation according to the choice of c_α. That is, to account for a change in the choice of c_α from the standard model choice of 1, one may simply multiply Ψ_n or Ψ on the left by $c_\alpha^{-0.5}$.

Unfortunately, the Dirac equation was chosen more for its ease of solution than for its ability to represent elementary particles. The linearity of the Dirac equation is somewhat incompatible with the requirements of the Pauli exclusion principle. The standard model is able to get around this difficulty through the use of configuration space, but for a fundamental theory, a nonlinear wave equation is more useful. Accordingly, we will next generalize our results of this section to nonlinear wave equations.

II. NONLINEAR WAVE EQUATIONS

As an example of a nonlinear wave equation, consider:

$$\partial_t\,\Psi = c_\alpha \nabla(\Psi^2), \quad (21)$$

As before, we can commute $c_\alpha^{+0.5}$ around ∇ to get:

$$c_\alpha^{-0.5}\;\partial_t\,\Psi = \nabla(c_\alpha^{-0.5}\Psi^2), \quad (22)$$

If we define $\Psi' = c_\alpha^{-0.5}\Psi c_\alpha^{+0.5}$, we can put this equation into the following simple form:

$$\partial_t\,\Psi' = \nabla(\Psi'^2), \quad (23)$$

Thus we can use our generalized speed of light in nonlinear wave equations as well.

Note that the above example indicates that a more general method of taking account of the effect of c_α is the transformation:

$$\Psi' = c_\alpha^{-0.5}\Psi c_\alpha^{+0.5}. \quad (24)$$

It is clear that the above transformation is compatible with both linear and nonlinear equations. It is also clear that this transformation will provide isomorphisms for groups based on multiplication of elements of a GA such as the "Lounesto groups" of the previous paper.[9] For the remainder of this paper, we will stick to the example of the Dirac equation. The nonlinear equation Eq. (21) will be the subject of a later paper, but we will use this transformation for transforming functions for the remainder of this paper.

III. PROPAGATORS AND PARTICLES

In the standard model of the elementary particles, the identification of the elementary fermions is by a method separate from the Dirac equation itself. That is, the same Dirac equation suffices for each elementary fermion, with the same equation shared by all the particles. We have previously introduced[9] a more general Dirac equation that can be separated into individual Dirac equations, one for each particle type.

This concept of a single Dirac equation whose ideals correspond to the various particles is somewhat at variance with the use of configuration space in quantum mechanics. This issue will be addressed in a later paper. For the purposes of this paper, it suffices to keep the configuration space concept that all particles of a particular type use the same wave equation (and therefore share the same form of propagator), but at the same time to reject the apparent coincidence that particles of distinct types happen to share an identical wave equation.

Another way of putting this is to recognize that, the fact that the GA allows us to write a multiple representation of the Dirac equation, implies that we can associate each of the contained represenations with a particular particle type, and use that representation when propagating that type of particle. Since the solutions are isomorphic, there is no actual difference in practical calculations. The various particles are still being propagated according to the same simple Dirac equation, it's only that we will remember that we actually have a multiple representation of the Dirac equation, so we can associate a different representation of the Dirac equation with each particle type.

This method of associating the propagators with a representation that depends on the particle type is a way of introducing geometric principles to quantum field theory with the least amount of change to the traditional technique, so we will use it for the remainder of this paper.

IV. C, P, T AND θ_W

In a previous paper[9], geometric derivations of the operators corresponding to spin, charge, parity and time were posited to be:

$$\begin{aligned} \tau_{Sz} &= \widehat{ixy}/2, \\ \tau_t &= \hat{t}/2, \\ \tau_p &= \hat{s}/2, \\ \tau_c &= \widehat{st}/2. \end{aligned} \qquad (25)$$

Since $\hat{t}$ and $\hat{i}$ ares only notational vectors, they commute with c_α. Thus, from the point of view of the geometrical elements, the first two of the above operators are even and therefore commute with c_α, while the second two are odd and inverse commute. Accordingly, we can compute how these operators transform by Eq. (24):

$$\begin{aligned} \tau'_{Sz} &= \tau_{Sz}, \\ \tau'_t &= \tau_t, \\ \tau'_p &= \tau_p c_\alpha = c_\alpha^{-1}\tau_p, \\ \tau'_c &= \tau_c c_\alpha = c_\alpha^{-1}\tau_c. \end{aligned} \qquad (26)$$

The above is not surprising in that it is known that the weak force violates only parity and charge symmetry in a "maximal" way. Time reversal symmetry $T = CP$ is also violated, but at a very small rate.[10, §4.12]

The lowest order diagrams contributing to T symmetry violation require flavor mixing among all three fermion families, and are therefore suppressed both by the small size of the off diagonal Cabibo-Kobayashi-Maskawa (CKM) matrix elements and the high order of the diagram.[3, §22.7] A later paper devoted to fermion masses will discuss the CKM matrix in the context of the PTG.

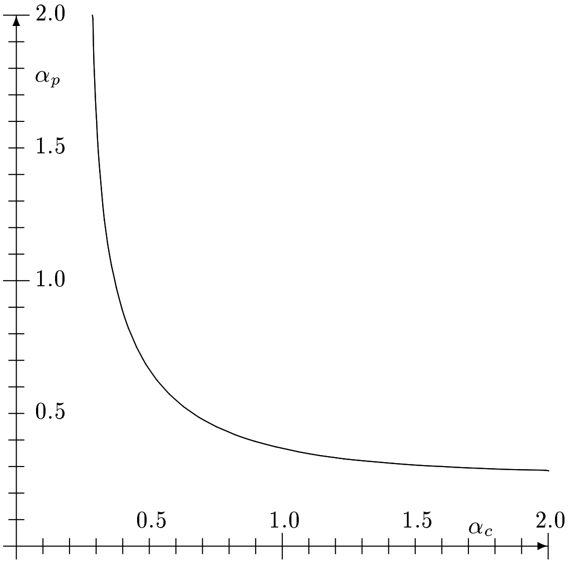
At low energies, the weak force is weak, compared to the electromagnetic force, almost entirely due to the mass of the exchange bosons associated with the weak force. The subject of fermion and boson masses will be discussed, along with the CKM matrix, in a later paper. But even ignoring the $W^\pm$ and Z^0 masses, the weak coupling constant is slightly weaker than the electromagnetic coupling constant. In the standard model, the difference is described by the Weinberg angle (sometimes called "weak mixing angle"), θ_W. We will now derive a restriction on the values of α_p and α_c based on the relative strengths of the weak and electromagnetic couplings. We will use $\sin^2(\theta_W) = 1/4$.

The operators for electromagnetic charge, Q_e, and weak charge, Q_w were derived [9] as:

$$\begin{aligned} \tau_{Qe} &= -(\hat{s} + \widehat{st}\,)/4, \\ \tau_{Qw} &= -(\hat{s} - \widehat{st} - 2\widehat{ixyt}\,)/4. \end{aligned} \qquad (27)$$

The $\widehat{ixyt}$ term is the operator that distinguishes between left and right handed versions of the same particle. It is therefore related to mass and its use will be delayed to a

FIG. 1: Relationship of α_p and α_c to give $\sin^2(\theta_W) = 1/4$.



later paper. Applying Eq. (24) to Eq. (27) gives:

$$\begin{aligned} \tau'_{Qe} &= \tau_{Qe} c_\alpha, \\ &= -((c_p + c_c)\hat{1} + (c_p + c_c)\hat{t} \\ &\quad +(c_1 + c_t)\hat{s} + (c_1 + c_t)\widehat{st}\ )/4 \qquad (28) \\ \tau'_{Qw} &= \tau_{Qw\sqrt{3}} c_\alpha, \\ &= -((c_p - c_c)\hat{1} - (c_p - c_c)\hat{t} \\ &\quad +(c_1 - c_t)\hat{s} - (c_1 - c_t)\widehat{st}\ )/4. \qquad (29) \end{aligned}$$

To obtain the a restriction on α_p and α_c, use Eq. (12) to obtain:

$$\begin{aligned} \tau'_{Qe} &= -(\sinh(\alpha_p + \alpha_c) + \cosh(\alpha_p + \alpha_c)\hat{s}\,)(\hat{1} + \hat{t}\,)/4, \\ \tau'_{Qw\sqrt{3}} &= -(\sinh(\alpha_p - \alpha_c) + \cosh(\alpha_p - \alpha_c)\hat{s}\,)(\hat{1} - \hat{t}\,)/4. \end{aligned} \qquad (30)$$

The electromagnetic charge Q, is therefore multiplied by a factor of $\cosh(\alpha_p + \alpha_c)$, while the weak charge Q' is multiplied by $\cosh(\alpha_p - \alpha_c)$. In order for these two factors to give an increase in the electromagnetic coupling, relative to the weak coupling, by a factor of $\cot(\theta_W) = \sqrt{3}$, the restriction on α_p and α_c is:

$$\cot(\theta_W) = \cosh(\alpha_p + \alpha_c)/\cosh(\alpha_p - \alpha_c). \qquad (31)$$

For $\sin^2(\theta_W) = 1/4$, the resulting restriction on α_p and α_c is shown in Fig. (1).

[1] C. A. Brannen, *The proper time geometry*, `http://www.brannenworks.com/a_ptg.pdf` (2004).
[2] D. Hestenes, *New Foundations for Classical Mechanics* (Kluwer Academic Publishers, 1999).
[3] I. J. R. Aithison and A. J. G. Hey, *Gauge Theories in Particle Physics, Volume II* (Institute of Physics Publishing, 2004).
[4] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill Book Company, 1964).
[5] A. Zee, *Quantum Field Theory In A Nutshell* (Princeton University Press, 2003).
[6] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley Publishing Company, 1995).
[7] J. M. C. Montanus, Found. Phys. **31**, 1357 (2001).
[8] J. B. Almeida, *The null subspaces of g(4,1) as source of the main physical theories*, `http://www.arxiv.physics/0410035` (2004).
[9] C. A. Brannen, *The geometry of fermions*, `http://www.brannenworks.com/a_fer.pdf` (2004).
[10] B. P. Roe, *Particle Physics at the New Millennium* (Springer-Verlag, 1996).