The Geometry of Fermions

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This paper analyzes the structure of the elementary fermions using the Geometric Algebras derived from several candidates for the manifold of space-time. One candidate, the Proper Time Geometry, is shown to be consistent with a simple interpretation of the fermions that requires subparticles here called "binons." An explicit solution for the fermion structure is shown. The result is a fully geometric version of the standard model particle system.

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This paper describes geometry of the fermions from the point of view of the Geometric Algebras (GA) of serveral candidate manifolds. In addition to the standard Lorentz-Minkowski Geometry (LMG) and the Proper Time Geometry (PTG), the mathematical machinery necessary to analyze other manifolds is developed. The PTG is an alternative geometry for Special Relativity (SR) that shares the same local metric equation as the LMG, but a different interpretation of the coordinates, and a different global topology.[1]

The first section provides a brief and intuitive introduction to the GA of David Hestenes, which, technically, are Clifford Algebras whose vector bases are the tangent spaces to the manifold with a metric chosen to match that of the manifold. [2, Chap. 1-2] The second section discusses the importance of primitive idempotents in quantum mechanics, and shows the structure of the primitive idempotents of the GAs of the LMG and PTG.

The third section analyzes the structure of the quantum numbers of the elementary fermions in light of the results of the first two sections, and the attributes of a compatible GA are deduced. Here it is shown that the elementary fermions are not likely to be elementary, but instead are each composed of three sub particles, and that a good candidate for the space-time manifold is the Proper Time Geometry.

The fourth section derives the (intrinsic) geometric operators corresponding to the discrete symmetries of charge conjugation, parity negation, and time reversal (C, P, and T). The fifth section demonstrates the $\mathcal{SU}(3) \times \mathcal{SU}(2) \times \mathcal{U}(1)$ symmetry of the elementary fermions. The doublet and dual singlet representation structure of $\mathcal{SU}(2)$ is derived. The sixth section derives the $\mathcal{SU}(3)$ symmetry, the binon binding potential, and the fermion spatial wave functions.

I. THE GEOMETRIC ALGEBRA (GA)

This paper takes the point of view that the elementary particles are vibrations inherent to some media that makes up space-time, that space-time can be modeled as a manifold, and that the vibrations of space-time can be modeled with the GA associated with that space-time manifold. The objective is to provide an ontological model for the fermions that requires fewer and simpler assumptions that that of the standard (phenomenological) model.

I chose to use the GA in this endeavor for several reasons. First, there is a good history of success in modeling physics with GAs. [2] Second, use of GAs is not widespread, so it is more likely to have undiscovered pleasant surprises. Third, there are no unphysical degrees of freedom in a GA, which makes them more compatible with the derivations of gauge theory. In other words, since this is to be an ontological model, the fact that the GA does not require the use of quotient spaces makes the ontological interpretation much simpler. Fourth, the fact that a GA is a Clifford Algebra defined on a manifold allows the extensive machinery of Clifford Algebras to be brought to bear. Fifth, the GA provides a natural description of displacements of a continuum in a way that is similar to the use of tensors in General Relativity, but without the unphysical degrees of freedom. Since, at the time of this writing, the GA is not widely used, we will begin with a brief and intuitive introduction.

A GA is a generalization of the algebra of real numbers (scalars). By "algebra" is meant a collection of "numbers" with rules indicating how to add and multiply amongst them. The elements of the GA are particularly suited to describing small deformations in a media. In the case of this paper, the small deformations are those of space-time itself. If one interprets a manifold of n-dimensions as a media subject to deformations, then one can model the deformations that remain within the media (as opposed to the situation of a media present in a higher dimensional space that can deform into dimensions not present in the media) using the GA. Since these are small deformations, they are linear in that one can consider adding two deformations and the result can also be interpreted as a deformation. The deformations therefore form a vector space. As numbers, elements of a GA are elements of a Clifford Algebra. The standard designation of Clifford Algebras is $\mathcal{CL}^{p,q}$ where p is the number of positive signature basis elements and q is the number of negative signature elements. The sum, $p+q$ therefore

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corresponds to the total dimension of the manifold associated with the GA. Since a GA is associated with a particular manifold, it forms a natural field of numbers for functions defined on that manifold. It is this natural use of the GA that forms the basis for its use in quantum field theory.

The 2^n basis elements for a GA can be labelled as the 2^n distinct subsets of a set of n basis vectors of the tangent space of the underlying manifold. As an example, consider the "Proper Time Geometry" (PTG) a manifold of 4 dimensions with signature $++++$. The elements of the GA at a point in the manifold depend only on the dimension and metric signature of the manifold, so if we are interested only in the local properties of the GA, we can specify this GA as $GA(++++)$, or use the Clifford Algebra designation of $\mathcal{CL}^{4,0}$. A third notation would be $GA(PTG)$ or if we wish to take special note of the coordinates used in the manifold, $GA(xyzs)$.

In the case of the $GA(PTG)$ the underlying manifold is of dimension 4, so the $GA(PTG)$ has dimension $2^4 = 16$. We will use the following labels, derived from the standard coordinate names, for the sixteen (canonical) basis vectors of GA(PTG):

$$\{\hat{1}, \hat{x}, \hat{y}, \hat{z}, \hat{s}, \widehat{xs}, \widehat{ys}, \widehat{zs}, \widehat{xy}, \widehat{yz}, \widehat{xz}, \widehat{yzs}, \widehat{zxs}, \widehat{xys}, \widehat{xyz}, \widehat{xyzs}\}. \quad (1)$$

The canonical basis vectors are of particular importance in the GA. They can be interpreted as deformations of the underlying manifold. Of the above sixteen deformations, the first few are easiest to visualize. The $\hat{1}$ deformation is the scalar deformation and corresponds, when positive, to compression, and when negative, to rarefaction.

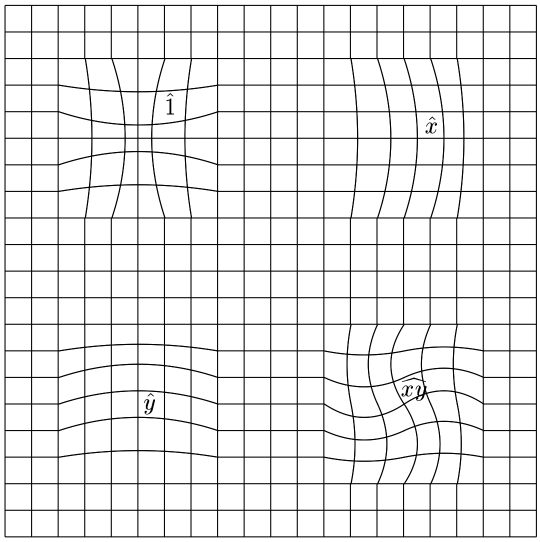
It is the standard in the literature to write $\hat{1} = 1$, as $\hat{1}$ is the multiplicative identity of the algebra, but I will rebel against this usage, at least in this paper, as I wish to make it clear that for this theory, $\hat{1}$ is a specific deformation of space or space-time, and not just a multiplicative identity. The absence of a deformation is 0, the additive identity of the algebra, and this I agree to write without the hat.

The vector deformations, $\hat{x}$,$\hat{y}$, $\hat{z}$, and $\hat{s}$ correspond to displacements in those directions. The bivector deformations, $\widehat{xs}$, $\widehat{ys}$, $\widehat{zs}$, $\widehat{yz}$, $\widehat{zx}$, and $\widehat{xy}$ are rotations defined in their associated planes.

The trivector or psuedovector deformations, $\widehat{yzs}$, $\widehat{zxs}$, $\widehat{xys}$ and $\widehat{xyz}$ are difficult to visualize, but act like axial vectors. Finally, the psuedoscalar deformation $\widehat{xyzs}$ is similar to the psuedoscalars used elsewhere in physics. It is the only deformation that is handed, in the sense that its mirror image cannot be brought into congruence with the original deformation through a rotation.

The deformations are defined in the sense of modifications to the metric of space-time. They are not intended to be descriptions of literal displacements as if space-time were embedded in some higher dimensional space. If there were an embedding, one might suppose that it could be modeled with the four vector deformations only.

FIG. 1: Four deformations of space are shown. The scalar deformation $\hat{1}$ is in density. The vector deformations $\hat{x}$ and $\hat{y}$ move space-time in their respective directions. The bivector deformation $\widehat{xy}$ rotates space-time in the $x-y$ plane.



With this caveat in mind, it may be useful to obtain an intuitive feel for the canonical basis elements by examining Fig. (1), which shows the deformations of the $x-y$ plane.

Addition of GA elements is as with any vector space. The basis notation helps as a reminder of how to multiply, in that, for example, $\hat{x}\hat{y} = \widehat{xy}$. Multiplication is associative, but not commutative, and the square of basis vector element is, depending on the signature, either 1 or -1. For the $GA(PTT)$ the signature is positive so the squares of basis elements are all unity: $\hat{x}\hat{x} = \hat{1}$. The product of two distinct vector basis elements is anticommutative: $\hat{x}\hat{y} = -\hat{y}\hat{x}$. This can be thought of as a reversal of the direction of rotation defined by the bivector. These rules, along with the usual distribution of multiplication over addition, are sufficient to define multiplication of arbitrary elements of a GA.

Note that in a GA with signature of x and y of $(++)$ or $(--)$, we will have $\widehat{xy}^2 = -1$. Elements that square to -1 can be thought of as geometric equivalents of imaginary numbers. There are many examples where imaginary numbers in standard physics equations have been replaced with GA elements, and the literature gives one a feeling that most of the small group of physicists using GA eschew imaginary numbers having no geometric interpretation. This paper will follow this trend. In this paper, compactification will provide a geometric meaning to the complexified geometries.

For the reader unaccustomed to the GA or Clifford Algebras, a few calculational examples, with the signature of $(++++)$ will assist in reading this paper, as well as provide motivation for the symmetry calculations done later.

$$
\begin{aligned}
(\hat{x}+i\hat{y})^2 &= 0(\text{nilpotent}) \\
(0.5+0.5\hat{x}\,)^2 &= (0.5+\hat{x}\,)(\text{idempotent}) \\
\widehat{xy}^{\,2} &= -1 \;\; (\text{rep. of}\,\mathcal{C}), \\
\{i\widehat{yz}, -i\widehat{xz}, i\widehat{xy}\} & \quad\;\; (\text{rep. of}\,\mathcal{SU}(2))
\end{aligned}
$$

An operation called "reversion" is sometimes useful. Reversion is indicated with a dagger and reverses the order of multiplication: $(AB)^\dagger = B^\dagger A^\dagger$. Reversion has no effect on scalars, vectors or psuedoscalars, but it has the effect of negating bivectors and trivectors. Note that if a multivector can be factored into vectors, its product with its reverse is a scalar, and one can therefore define an inverse for such elements. But as can be seen from the examples, not all elements of a GA possess a multiplicative inverse.

David Hestenes has applied the GA to the Dirac equation, and has found that the equation and its wave solutions can be written in the GA with some insight into their geometric properties. His work is defined in a variety of a GA called the "Space Time Algebra" (STA). The STA is obtained by using four unit vectors for the usual four directions in special relativity, and assuming that the square of the basis vector is positive in the case of time and negative in the cases of the three spatial dimensions. We can designate this GA as $GA(STA)$, $GA(+---)$, $GA(txyz)$, or $\mathcal{CL}^{1,3}$.

Hestenes' analysis is based on the Dirac equation. The $\{\gamma^\mu\}_{\mu=0}^3$ matrices are reinterpreted as unit vectors in the STA. The characteristic equation for the gamma matrices, $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = g^{\mu\nu}$, is satisfied when the signature of the associated GA is chosen to match. Unfortunately, the Dirac equation describes a complicated mixture of a collection of very distinct fermions. Here "distinct" is used in the sense of distinguishable. For example, when the Dirac equation is used to describe an electron, there are, in fact, from the point of view of the chiral elementary particles, four very different particles involved, right and left handed electrons, and right and left handed positrons.

This paper will take the position that the Dirac equation is not a fundamental feature of the underlying field theory, but is instead simply the result of a renormalization or resummation of the propagators of a set of four distinct fermions, as will be discussed in a later paper. Accordingly, this paper will analyze Hestenes' STA, but from the point of view of the chiral wave states, rather than the Dirac equation.

II. PRIMITIVE IDEMPOTENTS

Following Julian Schwinger's 1955 lectures on quantum kinematics, but specializing to the case of the electron fermion family, [3, Chap. 1.1] let A denote a set of characteristics that distinguish the 32 particles in a family $\mathcal{F}$ of elementary fermions. We will define the QCD colors as $\{1,2,3\}$, and use the following designations for such a family:

$$
\begin{aligned}
\mathcal{F} = \quad & \{e_L, e_R, \bar{e}_R, \bar{e}_L, \nu_L, \nu_R, \bar{\nu}_R, \bar{\nu}_L, \\
& u_{1L}, u_{1R}, \bar{u}_{1R}, \bar{u}_{1L}, u_{2L}, u_{2R}, \bar{u}_{2R}, \bar{u}_{2L}, \\
& u_{3L}, u_{3R}, \bar{u}_{3R}, \bar{u}_{3L}, d_{1L}, d_{1R}, \bar{d}_{1R}, \bar{d}_{1L}, \\
& d_{2L}, d_{2R}, \bar{d}_{2R}, \bar{d}_{2L}, d_{3L}, d_{3R}, \bar{d}_{3R}, \bar{d}_{3L}\}
\end{aligned}
$$

Let a_1 be an elementary particle in $\mathcal{F}$. Let $M(a_1)$ symbolize the selective measurement that accepts particles of type a_1, and rejects all others. One can imagine some sort of Stern-Gerlach apparatus, though since quarks are permanently bound it will have to be an imaginary apparatus. We can define addition of measurements to be the less selective measurement that accepts particles of any of the included types:

$$M(a_1) + M(a_2) = M(a_1 + a_2). \tag{2}$$

Two successive measurements can be represented by multiplication of the measurement symbols. Because of the physical interpretations of the symbols, addition is associative and commutative, while multiplication is at least associative. One and zero represent the trivial measurements that accept all or no particles. Clearly, $0 + M(a^1) = M(a^1)$, $1M(a^1) = M(a^1)1 = M(a^1)$, and $0M(a^1) = M(a^1)0 = 0$, so the set of measurements form an algebra. The "elementary" measurements associated with these 32 fermions satisfy the following equations:

$$M(a^1)M(a^1) = M(a^1), \tag{3}$$

$$M(a^1)M(a^2) = 0, \text{ if } a^1 \neq a^2, \tag{4}$$

$$\sum_{n=1}^{32} M(a^n) = 1 \tag{5}$$

Schwinger goes on to analyze incompatible measurements, such as spin in two different directions, but these simple results are enough for our purposes.

The repetition of the elementary particles in various families suggests that the higher families are simply excited states of the electron family. Since the excited states of standard quantum mechanics are formed by differences in spatial waveforms, rather than changes to the field of which the wave is composed, we naturally assume the contrapositive, and suppose that the differences between the fermions of a single family are due only to non spatial differences in their wave functions. With this natural assumption, the 32 elementary fermions should be able to be distinguished by elements of the GA rather than by spatially distinct wave functions. That is, we should be able to use the Clifford Algebra to describe the electron family, rather than require the use of the full GA wave functions as defined on the manifold.

The assumption that the elementary fermions of any family can be described by the Clifford Algebra of the space-time manifold, without requiring an understanding of their wave functions, considerably simplifies the task of deriving their structure. The mathematicians having already solved these problems, we shall define the concepts of the Measurement Algebra in their language. When an element of an algebra is its own square, as in Eq. (3), the mathematicians refer to it as an "idempotent". When two idempotents of an algebra multiply out to zero as in Eq. (4), they are called "perpendicular".

The measurements here are assumed elementary in the sense that the elementary particles they describe cannot be broken (we assume) into subparticles. This corresponds to the concept of an idempotent that cannot be written as the sum of two perpendicular idempotents, which the mathematicians call a "primitive idempotent". Finally, it is easy to prove that (complete) sets of perpendicular primitive idempotents for Clifford Algebras add to unity as in Eq. (5). Therefore, our assumption about the elementary fermion families requires that we examine the mathematical literature for information about sets of perpendicular primitive idempotents of Clifford Algebras.

A left (right) "ideal" $\mathcal{I}$ of an algebra $\mathcal{CL}$ is a subalgebra that is closed under multiplication by elements of the algebra on the left (right). That is, $\mathcal{CL}\ \mathcal{I} = \mathcal{I}$. A minimal ideals is one that has no nontrivial subideals. It is easy to show that a primitive idempotent e_n generates a minimal left ideal $I = \mathcal{CL}e_n$.

Ideals are particularly important in differential equations because they can be used to derive symmetry relations and reduce dimensionality of systems of differential equations. For example, as shown in the appendix, for GA valued functions defined on the PTG manifold, the Dirac equation can be derived from:

$$\partial_t \psi = \nabla \psi, \tag{6}$$

where ψ is a function on the space-time manifold that takes its values from the associated GA. If $\{e_n\}_{n=1}^N$ is a complete set of perpendicular primitive idempotents (as we are postulating to represent a family of elementary particles), then the above equation can be rewritten as N independent equations, each of the form:

$$\partial_t(\psi e_n) = \nabla(\psi e_n), \tag{7}$$

Thus a complete set of primitive idempotents gives a way of breaking a complex representation of the Dirac equation into individual representations, one for each elementary particle. This subject will be covered at length in a later paper[4].

Theorem (Lounesto [5]). A minimal left ideal of a Clifford Algebra $\mathcal{CL}^{p,q}$ is of the type $\mathcal{I} = \mathcal{CL}^{p,q}e$, where

$$e = \frac{1}{2}(1+e_1)...\frac{1}{2}(1+e_k) \tag{8}$$

is a primitive idempotent of $\mathcal{CL}^{p,q}$ and $\{e_n\}_{n=1}^k$ are a set of elements of $\mathcal{CL}^{p,q}$ such that

$$\begin{aligned} e_n e_m &= e_m e_n, \\ e_n &\in \text{Canonical basis}, \\ e_n^2 &= 1, \\ \{e_n\}_{n=1}^k & \quad \text{generates a group of order } 2^k, \end{aligned} \tag{9}$$

$k = q - r_{q-p}$ and r_i are the Radon-Hurwitz numbers, defined by the recurrence formula $r_{i+8} = r_i + 4$ and

i	0	1	2	3	4	5	6	7
r_i	0	1	2	2	3	3	3	3

In this paper, we will refer to a set of elements that satisfy Eq. (9) as "generators of the Lounesto group", and we will refer to the generated group as the "Lounesto group". As an example, the STA is equivalent to $\mathcal{CL}^{1,3}$ and the above formula gives $k_{STA} = 1$. A choice for e_1 is the time vector, $\hat{t}$, or a product of a space and the time vector such as $\widehat{zt}$. This gives a typical primitive idempotent of $\frac{1}{2}(1+\widehat{zt})$, and a complete set of primitive idempotents as $\{\frac{1}{2}(1+\widehat{zt}), \frac{1}{2}(1-\widehat{zt})\}$.

Unfortunately, these sets of primitive idempotents each include only two candidates, not nearly enough for the 32 fermions in a family. On the other hand, it is at least somewhat heartening that the fermion family does have a power of two elements. Accordingly, we now consider ways in which we could naturally increase the number of commuting elements in the canonical basis, and therefore the number of primitive idempotents.

Note that the above theorem was for a real Clifford Algebra. If we complexify the algebra, we will obtain more canonical basis elements that square to 1 so the theorem does not apply. Also note that the product of two primitive idempotents is also a primitive idempotent. One can therefore pick k elements out of the Louneseto group, and as long as the chosen elements generate the whole group, they can be used as an alternate set for the purposes of defining the primitive idempotents. We will later use this fact to simplify the structure of the fermions.

At this point, it is useful to introduce a modification of the usual notation for complexified Clifford Algebras. Instead of considering complex linear combinations of elements of the canonical basis for the real Clifford Algebra, we will instead double the size of the canonical basis by including imaginary canonical basis elements. For example, in addition to $\hat{x}$ as a basis element, we will include a basis element $\widehat{ix}$ whose properties are those of $i\hat{x}$. With this notation, $\hat{i}$ can be interpreted as a vector that commutes with all other basis elements (and therefore with all algebra elements), and that squares to -1.

Manifolds with a compactified (hidden) dimension, such as the PTG, can be complexified naturally by complex Fourier series over the hidden dimension, as is shown in the appendix. The 1st, 2nd and 3rd order terms in the Fourier series are interpreted as the electron, muon and tau families of fermions, and the 0th order term is interpreted as the bosons. This unites the propagators for all the particles into a single wave function.

The PTG does not include time in the geometry. It is therefore natural to suppose that a wave state in the PTG may require two GA fields, one for the displacement from neutral, the other for the rate of change or momentum of the displacement. For the purposes of this paper, this doubling of the field variables can be modeled with the addition of another commuting basis vector. Unlike the case with the STA, t has no role in the PTG, so we can use $\hat{t}$ to designate the momentum. This is an expansion of the basis elements similar to that of complexification. For example, the basis element $\widehat{xy}$ defines a rotation in the $x-y$ plane, while the basis element $\widehat{xyt}$ defines the time rate of change of the amount of rotation in the $x-y$ plane. Addition is as usual with vector spaces. For multiplication, the basis vectors $\hat{i}$ and $\hat{t}$ commute with everything else and square to ± 1.

In order to accustom the reader to the notation, it is useful to include a concrete example for eigenvector equations in the complexified STA. The complexified STA, as opposed to the standard STA, has pairs of canonical basis elements that square to one and also commute. For example, $\{\hat{t}, \widehat{ixy}\}$ are a set. This defines a complete set of $2^2 = 4$ primitive idempotents:

$$\{(1 \pm \hat{t})(1 \pm \widehat{ixy})/4\} \tag{10}$$

A set of Lounesto generators can be interpreted as a set of commuting operators with the primitive idempotents as eigenvectors. The eigenvalues are ± 1, which is somewhat disconcerting to a physicist accustomed to half-integer eigenvalues, so we will instead consider the operators to be halves of the commuting canonical basis elements. Per the example calculations, this has the beneficial effect of putting the operators into the standard form for Lie algebras. The resulting eigenvector equations for the $\tau_t = [0.5\hat{t}\,]$ operator are as follows:

$$\begin{aligned} \tau_t(1+\hat{t})(1 \pm \widehat{ixy}\,)/4 &= +0.5(1+\hat{t})(1 \pm \widehat{ixy}\,)/4, \\ \tau_t(1-\hat{t})(1 \pm \widehat{ixy}\,)/4 &= -0.5(1-\hat{t})(1 \pm \widehat{ixy}\,)/4. \end{aligned} \tag{11}$$

The eigenvector equations for the $0.5\widehat{ixy}$ operator are similar, but with signs for the eigenvalue in the series $+-+-$.

Following this example, we can connect the notion of idempotents and ideals with the usual physics terminology of eigenvector and eigenvalues Accordingly, we can abbreviate the designations of the eigenvectors. Given a set of Lounesto generators given by $\{e_n\}_{n=1}^{k}$, we will define the operators and eigenvectors as follows:

$$\tau_j = 0.5 e_j, \tag{12}$$

$$|n,...l,m>= (0.5+ne_1)...(0.5+le_{k-1})(0.5+me_k), \tag{13}$$

where $n,...l,m$ are the eigenvalues, and take values of $\pm 1/2$. The symmetry structure of the eigenvectors is that of a k-cube. Since the eigenvalues are either positive or negative $1/2$, a natural notation for the eigenvectors, for example for the 3-cube, is $\{|--- >, |--+ >, ...|+++ > \}$.

The value of k for real GAs is given by the Lounesto theorem. For a complexified GA of dimension n, the value of k is easily seen to be $n/2$ if n is even, and $(n+1)/2$ if n is odd, with primitive idempotents generated by a set of k elements with the same requirements of the Lounesto theorem, Eq. (9). For those manifolds that do not explicitly include time, an extra commuting operator (i.e. $0.5\hat{t}$) accounting for momentum versus position must be included and this increases the value of k by one.

We can now compute the value of k, and therefore the expected symmetry of a fermion family for various choices of the space-time manifold. Geometric canonical basis vectors with positive and negative signature, and notational canonical basis vectors, along with the k value are shown here:

Manifold	p(+)	q(-)	not.	k	Symmetry
STA	t	xyz		1	line
	xyz	t		2	square
PTG	xyzs		it	3	cube
		xyzs	it	3	cube
	xyzs	t	i	3	cube
	t	xyzs	i	3	cube

The largest value of k is for the PTG and similar geometries complexified through compactification, and the resulting symmetry is that of the cube, with 8 fundamental fermions. The electron family includes four times too many, but there are only 8 electron family leptons, and the next section will show that these do have a cubic symmetry.

The question of whether or not the "internal" symmetries of particles, other than spin, can be nontrivially connected to the "external" geometry of space-time has already been answered, in the negative, by various "no-go" theorems, most notably that of Coleman and Mandula. [6] These theorems all assume SR, and therefore perfect Poincaré symmetry. This is a subtle argument for rejecting any candidate manifold for space-time that mixes space and time. The PTG, in addition to assuming a preferred reference frame, explicitly rejects time as part of the geometry. Thus the Poincaré symmetry possessed by the PTG is not perfect. On the other hand, the Appendix shows that the simplest wave equation in the PTG is equivalent to a multiple rep of the Dirac equation, which is generally thought to be the standard for relativistic waves. In a certain sense, the PTG is in the same position as Quantum Field Theory. While the theory itself is not apparently Poincaré invariant, the results of computations are.

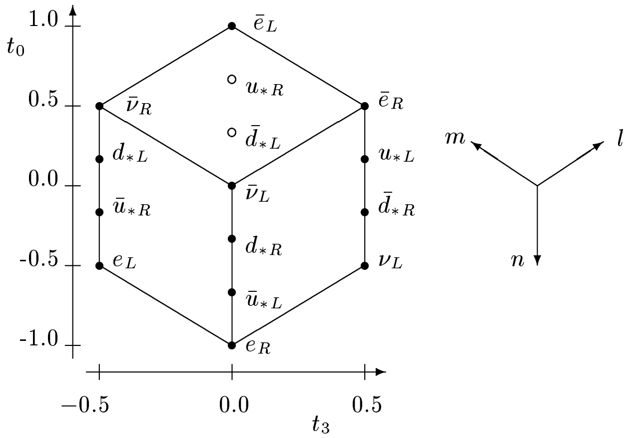
III. THE FERMION CUBE

The fermions in a family can be designated by their $\mathcal{SU}(2)$ and $\mathcal{U}(1)$ symmetry quantum numbers t_3 (weak isospin) and t_0 (weak hypercharge), or alternatively, by

FIG. 2: Table of standard model fermion quantum numbers.

	t_3	t_0	Q	$Q'\sqrt{3}/2$
e_R	0	-1	-1	1/2
e_L	-1/2	-1/2	-1	-1/2
ν_L	1/2	-1/2	0	1
ν_R	0	0	0	0
d_{*R}	0	-1/3	-1/3	1/6
d_{*L}	-1/2	1/6	-1/3	-5/6
u_{*L}	1/2	1/6	2/3	2/3
u_{*R}	0	2/3	2/3	-1/3

FIG. 3: The fermion cube. The ν_R is not shown for clarity.



their electric charge Q and "neutral charge" (or "weak charge") Q'. [7, Table 6.2] The values for Q and Q' are related to t_3 and t_0 by

$$\begin{aligned} Q &= t_3 + t_0, \\ Q' &= t_3 \cot(\theta_w) - t_0 \tan(\theta_w), \end{aligned} \tag{14}$$

where θ_w is the Weinberg angle. We will use $sin^2(\theta_w) = 1/4$. A table of the usual quantum numbers for fermions is shown in Fig. (2). Values for antiparticles are the negatives of the values shown. When the t_0 and t_3 numbers are plotted against each other, the result is clearly cubic as shown in Fig. 3

The figure of the elementary fermions makes clear that while there is an obvious cubic structure to the leptons, the quarks are intermediate to pairs of leptons along four parallel edges of the cube. For concreteness, we will define the cube according to the n, l, and m vectors as shown in Fig. (3). Both of the undetected neutrinos, ν_R and $\bar{\nu}_L$, end up at the origin, and we have to choose which goes with the visible "top" part of the cube and which is hidden. Since the rest of the top of the cube (i.e. $\{\bar{\nu}_R, \bar{e}_L, \bar{e}_R\}$) are all antiparticles, we will place the $\bar{\nu}_L$ with them. The n vector therefore runs in the direction from the $\bar{\nu}_L$ towards the e_R, the l runs towards the $\bar{e}_R$, and the m runs towards the $\bar{\nu}_R$.

Fig. (3) shows that the leptons do have a cubic structure and can be interpreted as primitive idempotents. According to the illustrated choice of n, l, and m, and using the order $|n, l, m>$, the assignments for the eight leptons are as follows:

$$\begin{aligned} &\bar{\nu}_L \equiv |--->, \quad e_R \equiv |+-->, \\ &\bar{\nu}_R \equiv |--+>, \quad e_L \equiv |+-+>, \\ &\bar{e}_R \equiv |-+->, \quad \nu_L \equiv |++->, \\ &\bar{e}_L \equiv |-++>, \quad \nu_R \equiv |+++> \end{aligned} \tag{15}$$

The leptons are thus associated with the primitive idempotents of a GA, but the presence of the quarks suggests that we can do better if we assume that the quarks and leptons are bound states of three subparticles each.

Since the designation for the primitive idempotents has the feel of binary numbering to it, I will call these subparticles "binons". The leptons correspond to bound states of three identical binons, so each lepton has the natural association with binons shown in Eq. (15). The quarks are mixed bound states of three binons, with mixtures only possible among binons that share l and m quantum numbers. With this assumption, binons form bound states among three particles that differ at most by the n quantum number. If the three binons are identical (presumably they differ in spatial wave state, or are related by rotation of elements in the GA), the wave state is a lepton, while the mixtures correspond to quarks. Defining the bound states by $|n_1 l_1 m_1, n_2 l_2 m_2, n_3 l_3 m_3>$, the fermions are obtained as follows:

$$\begin{aligned} \bar{\nu}_L &\equiv |---,---,---> \\ d_{1R} &\equiv |+--,---,---> \\ \bar{u}_{1L} &\equiv |---,+--,+--> \\ e_R &\equiv |+--,+--,+--> \\ \bar{e}_R &\equiv |-+-,-+-,-+-> \\ u_{1L} &\equiv |++-,-+-,-+-> \\ \bar{d}_{1R} &\equiv |-+-,++-,++-> \\ \nu_L &\equiv |++-,++-,++-> \\ \bar{\nu}_R &\equiv |--+,--+,--+> \\ d_{1L} &\equiv |+-+,--+,--+> \\ \bar{u}_{1R} &\equiv |--+,+-+,+-+> \\ e_L &\equiv |+-+,+-+,+-+> \\ \bar{e}_L &\equiv |-++,-++,-++> \\ u_{1R} &\equiv |+++,-++,-++> \\ \bar{d}_{1L} &\equiv |-++,+++,+++> \\ \nu_R &\equiv |+++,+++,+++> \end{aligned} \tag{16}$$

where the other quark colors are obtained by rotating the odd binon through the three positions.

In the standard model, the charges of the quarks do not depend on color, so given the assignment of binon quantum numbers, it is possible to derive the relations for t_3 and t_0 in terms of n, l and m. Taking account of Fig. (2) and Eq. (3), we get a matrix equation for t_3 and

t_0:

$$\begin{pmatrix} 0 & -1 \\ -1/2 & -1/2 \\ 1/2 & -1/2 \\ 0 & 0 \\ 0 & -1/3 \\ -1/2 & 1/6 \\ 1/2 & 1/6 \\ 0 & 2/3 \end{pmatrix} = \begin{pmatrix} 3/2 & -3/2 & -3/2 \\ 3/2 & -3/2 & 3/2 \\ 3/2 & 3/2 & -3/2 \\ 3/2 & 3/2 & 3/2 \\ -1/2 & -3/2 & -3/2 \\ -1/2 & -3/2 & 3/2 \\ -1/2 & 3/2 & -3/2 \\ -1/2 & 3/2 & 3/2 \end{pmatrix} M. \quad (17)$$

Solving this equation, and taking account of Eq. (14), we obtain the solution:

$$\begin{aligned} t_{b3} &= (l-m)/6, \\ t_{b0} &= (l+m-2n)/6, \\ Q_b &= (l-n)/3, \\ Q'_b\sqrt{3} &= (l+n-2m)/3. \end{aligned} \quad (18)$$

The subscript b is for binon, and signifies that these quantum numbers are for the binons rather than the fermions. The symmetry operators, on the other hand, have values 3x as large as these.

Since n, l and m were chosen so that the vector $n+l+m$ is perpendicular to the plane of the standard quantum numbers, the above equations can be written as differences of quantum numbers. This is a particularly interesting way to write geometric basis elements, and there is consequently an easy derivation of the $\mathcal{SU}(2)$ symmetry of the standard model, which we will demonstrate in (V).

IV. CHARGE, PARITY AND TIME

As can be seen from Eq. (18), this paper has, so far, been concerned with additive quantum numbers. But the Lounesto group generators commute, and since their eigenvalues are ± 1, they can also be associated with multiplicative quantum numbers. This suggests that we can unite the notation for additive and multiplicative quantum numbers if we redefine the multiplicative quantum numbers to fit the same relationship we have already derived for n, l, and m quantum numbers.

Since the Lounesto group generators carry eigenvalues of ± 1, their products do the same, so we can naturally associate each element of the Lounesto group with an operator, and define multiplication of the operators by the multiplication of the Lounesto group. We will refer to the operators (which carry eigenvalues of $\pm\frac{1}{2}$) by τ and the Lounesto group elements by e, with the appropriate suffices. For example,

$$\tau_{nlm} = e_{nlm}/2 = e_n e_l e_m/2 = 4\tau_n\tau_l\tau_m. \quad (19)$$

It is clear that not too much should be made of the n, l, m quantum numbers. Since all 7 nontrivial elements of the Lounesto group correspond to operators that have valid eigenvalues among the binons, rather than just e_n, e_l, and e_m, it is clear that our choise of the generators for that set was somewhat arbitrary. For example, the set $\{e_{nlm}, -e_{lm}, e_{nl}\}$, would also have worked as the generators of the Lounesto group, but would not have allowed the computation of the charge operators as a linear sum of these generators.

The fact that C, P, and T can be used as operators with quantum numbers in standard quantum mechanics suggests that we can find geometric elements of the GA that correspond to each, and that these elements will multiply to unity. Note that it is more usual to have CPT multiply to a phase factor, but that our fermion notation, with its explicit interpretation of $\hat{\imath}$ as a rotation operator in the hidden dimension s, does not have explicit phase factors. The phase factors will reappear when waves are considered in the full GA functions on the space-time manifold; there they will represent relative rotations in the s dimension. Here we are considering only Clifford Algebra elements that represent particles and ignoring their (x, y, z, s) position, and therefore also ignoring their phase.

The motivation for looking at P, C, and T as Lounesto group elements is the observation that C, P, and T commute and multiply to unity. This is exactly the group multiplication rule among a Lounesto group of order 4.

Note that the definition of the CPT operators here is slightly different from that usually used in physics. Here we are defining, for example, the τ_P operator as an operator that has eigenvalues of $\pm\frac{1}{2}$ according as the eigenfunction has an "intrinsic" parity of $+1$ or -1. The usual definition of the parity operator P is to have that it changes the parity of its operand. [8, Eq. (3.123)] Thus the corresponding eigenvalue relations, for this paper as compared to the standard use are:

$$\begin{aligned} \tau_P|\pm> &= (\pm/2)|\pm>, \text{(this paper)} \\ P|\pm> &= \eta_\pm|\pm>, \text{(standard)} \\ P|\pm>P &= \pm|\pm>, \text{(standard)} \end{aligned} \quad (20)$$

where $\eta_\pm$ is a possible phase. The C and T operators correspond to τ_C and τ_T in similar fashion. With this change, C, P and T are brought into the same form as the Lounesto operators. Since C, P, and T are to commute with the Schwinger particle measurements, we must have that C, P, and T correspond to operators in the Lounesto group.

Clearly spin in the z direction, S_z is also in the Lounesto group. Since it is known how spin transforms with rotations of the coordinates for the manifold, there is a natural choice for the spin operators:

$$\tau_{Sx} = \widehat{iyz}/2,\ \tau_{Sy} = -\widehat{ixz}/2,\ \tau_{Sz} = \widehat{ixy}/2, \quad (21)$$

The fact that S_z must be in the Lounesto group places a restriction on the remaining elements in that they must commute with S_z. The possible choices must be in the (complexified) subgroup generated by $\{\widehat{xy}, \hat{z}, \hat{s}, \hat{t}\}$. But any element that has a $\widehat{xy}$ factor will be transformed by coordinate rotations of the real spatial dimensions, and this is incompatible with the definitions of C, P, and T. Thus the possible choices for those elements of the Lounesto group that will correspond to P, C, and T are reduced to $\{\hat{1}, \hat{s}, \widehat{xyz}, \widehat{xyzs}\} \times \{\hat{1}, \hat{t}\}$. Any two of the elements in $\{\hat{s}, \widehat{xyz}, \widehat{xyzs}\}$ anticommute, so the possible choices for C, P, and T include the following alternatives:

$$\begin{aligned} \{\hat{s}, \widehat{st}, \hat{t}\} \quad & \text{I} \\ \{\widehat{ixyz}, \widehat{ixyzt}, \hat{t}\} \quad & \text{II} \\ \{\widehat{xyzs}, \widehat{xyzst}, \hat{t}\} \quad & \text{III.} \end{aligned} \tag{22}$$

Each of these includes a natural choice, $\hat{t}$, for T.

The first selection is interesting in that there is no mention of z. Since we are representing a chiral particle moving at nearly c [10] in the $+z$ direction, perhaps Lorentz contraction reduces deformations in that direction. The direction of travel in the s direction, $\widehat{st}$ can be used as C, that is, to distinguish between particles and antiparticles. Intrinsic parity is then defined as $\hat{s}$, in recognition of the concept that x, y, z, s form a manifold and so, at least locally, parity in (x, y, z) can be obtained by rotation in (x, y, z, s), leaving s negated. This alternative gives a particularly simple form for the coupling to photons, and it will be the one used for the remainder of this paper.

Internal 3-d parity, in the form of $\widehat{ixyz}$ is explicitly included in the second choice, but this alternative would leave the internal binons without any explicit use of $\hat{s}$. In addition, this alternative includes imaginary numbers. While the fermions have a geometric interpretation of $\hat{\imath}$, according to the Fourier series complexification process the bosons do not, and we would like C, P, and T to apply to bosons as well as fermions. The third choice is the only one that includes a 4-d chiral element of the geometry. Neither of the other two have such an element even in their full Lounesto groups including S_z.

With the selection of the first alternative, the choices for C, P, and T give:

$$\begin{aligned} \tau_{Sz} &= e_{Sz}/2 = \widehat{ixy}/2, \\ \tau_C &= e_C/2 = \widehat{st}/2, \\ \tau_P &= e_P/2 = \hat{s}/2, \\ \tau_T &= e_T/2 = \hat{t}/2. \end{aligned} \tag{23}$$

The full Lounesto group is then $\{\hat{1}, e_{Sz}\} \times \{\hat{1}, e_C, e_P, e_T\}$.

In order to assign specific geometric elements to the binons, we must define their quantum numbers with respect to e_{Sz}, e_P, and e_C. We can then use our geometric definitions, Eq. (23), to find e_n, e_l, and e_m, and then to define τ_3, τ_0, Q, and Q' using Eq. (18). In order to do this, we must decide on the multiplicative quantum numbers for the elementary particles.

Particles should carry a positive τ_C eigenvalue, antiparticles a negative. The intrinsic parity for antiparticles is known to be the negative of that of the particles, but are otherwise unknown. If we assign the same parity to the electron and neutrino τ_P and τ_C will carry identical quantum numbers, so we choose the electron to have positive parity and the neutrino to have negative. The quantum number for τ_T is then determined by $CPT = 1$. The resulting quantum numbers are shown in Fig. (4). Solving for the S_z, C, P, and T in terms of e_n, e_l and e_m gives:

FIG. 4: Table of binon/lepton multiplicative quantum numbers.

	nlm	S_z	$C\ P\ T$
$\bar{\nu}_L$	$-\,-\,-$	$-$	$-\,+\,-$
$\bar{\nu}_R$	$-\,-\,+$	$+$	$-\,+\,-$
$\bar{e}_R$	$-\,+\,-$	$+$	$-\,-\,+$
$\bar{e}_L$	$-\,+\,+$	$-$	$-\,-\,+$
e_R	$+\,-\,-$	$+$	$+\,+\,+$
e_L	$+\,-\,+$	$-$	$+\,+\,+$
ν_L	$+\,+\,-$	$-$	$+\,-\,-$
ν_R	$+\,+\,+$	$+$	$+\,-\,-$

$$\begin{aligned} e_{Sz} = e_{nlm}, \quad & e_P = -e_l, \\ e_C = e_n, \quad & e_T = -e_{nl}. \end{aligned} \tag{24}$$

Solving for e_n, e_l and e_m gives:

$$\begin{aligned} e_n &= e_C = \widehat{st}, \\ e_l &= -e_P = -\hat{s}, \\ e_m &= -e_{Sz}e_T. = -\widehat{ixyt}. \end{aligned} \tag{25}$$

Using Eq. (18), the additive quantum operators, after multiplying the right hand side by 3 to account for the composite nature of fermions and dividing by 2 to account for the conversion from e to operator, are calculated as:

$$\begin{aligned} \tau_3 &= -(\hat{s} - \widehat{ixyt}\,)/4, \\ \tau_0 &= -(\hat{s} + \widehat{ixyt} + 2\widehat{st}\,)/4, \\ \tau_Q &= -(\hat{s} + \widehat{st}\,)/4, \\ \tau_{Q'\sqrt{3}} &= -(\hat{s} - \widehat{st} - 2\widehat{ixyt}\,)/4. \end{aligned} \tag{26}$$

The role of the hidden dimension, s, in coupling to exchange force bosons is clear. Also note that the coupling to the photon is particularly simple. Presumably this fact is associated with the masslessness of the photon.

V. FERMION SYMMETRY

There exists in the literature at least one interesting attempt to place the standard model $\mathcal{SU}(3) \times \mathcal{SU}(2) \times \mathcal{U}(1)$ symmetry into a GA. [9] That attempt, however, has certain disadvantages compared to this paper. First, it uses a complexified version of the STA without any physical justification for complexification. Second, it fails to identify any particle states. Third, while it locates the symmetry, it fails to show that the particular represenations, for example the singletons of $\mathcal{SU}(2)$, are naturally found in the STA. Fourth, it says nothing of why fermions come in families. In short, it demonstrates the symmtery, not the individual particles. GAs are equivalent to matrix algebras, so that it is possible to find the standard model algebra, using matrix methods, in a GA is not too surprising.

Given the geometric version of the τ_3 generator of weak isospin $\mathcal{SU}(2)$, we can now derive a geometrical form for the other two generators. First, we need the geometric form for total weak isospin. Examining Fig. (2), it is clear that the total weak isospin operator must return 3/4 on $\bar{\nu}_R$, $\bar{e}_R$, e_L, and ν_L, and zero on the other binons. From Eq. (15), we see that e_{lm} returns 1 on binons with total isospin of zero, and -1 on binons with total isospin of 3/4. Therefore, the total weak isospin operator is given by:

$$\tau^2 = 0.375(\hat{1} - e_{lm}) = 0.375(\hat{1} - \widehat{ixyt}\,). \qquad (27)$$

Note that $(\tau_3)^2 = 0.125((\hat{1} - \widehat{ixyt})$, so we will look for τ_2 and τ_1 to square to this same value. Also note that $e_\tau = 0.5(\hat{1} - \widehat{ixyt}\,)$ is a projector. This operator projects out the doublet part of the subalgebra. The perpendicular projector, $0.5(\hat{1} + \widehat{ixyt}\,)$ projects out the singlet part of the subalgebra. The dimensions of these two subalgebras are clearly identical, thus the doublet and dual singlet form for weak isospin.

To find τ_1 and τ_2, first note that these elements will have to be in the ideal generated by τ^2. Inside that ideal, we will have a spin$-1/2$ irrep of $\mathcal{SU}(2)$. The Pauli spin matrices therefore provide a clue in that σ_3 anticommutes with the other spin matrices. Consequently, we list the canonical basis elements that anticommute with τ_3, and look for a linear combination that squares to $(\tau_3)^2$. A solution is:

$$\begin{aligned} \tau_1 &= (\hat{x} - \widehat{iyst}\,)/4, \\ \tau_2 &= -(\widehat{ixs} + \widehat{yt}\,)/4, \\ \tau_3 &= -(\hat{s} - \widehat{ixyt}\,)/4. \end{aligned} \qquad (28)$$

There are various other solutions, for example, one can multiply both τ_1 and τ_2 by $\hat{t}$ to get a form where $\hat{y}$ instead of $\hat{x}$ appears alone, but there are no solutions that treat x and y on equal footing. This suggests that any isospin mixed state particles cannot be rotationally symmetric about their spin axis. The third choice for the CPT operators in Eq. (22) gives forms for τ_1 and τ_2 that are rotationally symmetric.

VI. BINON BOUND STATES

Without knowing the spatial waveforms for how binons are bound together to produce fermions, it is not possible to explicitly derive the $\mathcal{SU}(3)$ symmetry. But under the assumption that the interaction can be modeled as a pairwise potential, the presence of an $\mathcal{SU}(3)$ symmetry can be argued from the discrete symmetries that apply. Let r_{23}, r_{31} and r_{12} represent the distances between the three binons. Due to Lorentz contraction, one expects that the three binons will be in the same plane perpendicular to the direction of propagation. Rigid rotations of the three particles in that plane correspond to a $\mathcal{U}(1)$ symmetry. Perhaps this has something to do with weak hypercharge.

In addition to distances between the binons, one must also specify their relative positions in the hidden dimension s. Since s is a cyclic dimension, a natural way of describing the relative positions of the binons is by using their separation in 3-space as the magnitude of a complex number and their angular separation S_{ij} in s as the phase of the complex number. The three complex numbers can be used as a set of canonical coordinates for the binon system. The canonical coordinates are a vector of three complex values, which is a suitable object for application of $\mathcal{SU}(3)$ symmetries with a triplet representation:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} r_{23}\, e^{is_{23}} \\ r_{31}\, e^{is_{31}} \\ r_{12}\, e^{is_{12}} \end{pmatrix}. \qquad (29)$$

Under the assumption that the binding potential satisfies an $\mathcal{SU}(3)$ symmetry in the above canonical coordinates, we have derived that the binding potential can depend only on the squares of the magnitudes of the three complex numbers, and therefore on the distances between the binons (and not on their relative phases):

$$V_b(r_{12}, ...s_{23}) = V_b((r_{23})^2 + (r_{31})^2 + (r_{12})^2). \qquad (30)$$

The assumption that the binding potential can be written as a sum of pairwise potentials implies that the form of those potentials is that of a linear harmonic oscillator:

$$V_b(r_{12}, ...s_{23}) = ((r_{23})^2 + (r_{31})^2 + (r_{12})^2)V_0, \qquad (31)$$

where V_0 is a suitable constant. Thus the binding force between binons is linear, as is suspected of the gluon force between quarks, and binons are permanently bound into fermions. With all three binons identical, the bound state is evidently a singlet, but with one binon distinct, the degeneracy is broken to show the $\mathcal{SU}(3)$ symmetry as a triplet of colored particles.

The binons that make up a lepton are identical in the sense that they all belong to the same ideal, but they need not be identical with respect to their degrees of freedom within that ideal, nor in their spatial (i.e. x, y, and z) dependency. Since we have ideals with a geometric interpretation, we can study what the degrees of freedom within those ideals are. This will give an interesting explanation for why certain pairs of binons can mix to form the quarks. Accordingly, we multiply out the ideals and display them paired according to their ability to mix to form quarks. This is shown in Fig. (5), and it is clear that the mixing rule has to do with the notational degrees of freedom, $\hat{i}$ and $\hat{t}$. When one notes that the degrees of freedom that are not frozen out by the ideals is the subalgebra generated by $\{\hat{x}, \hat{y}, \hat{z}\}$, it becomes clear that the binons that mix are able to have identical geometric (as opposed to notational) degrees of freedom. In addition,

FIG. 5: Table of binon ideals. Quarks are mixtures formed from each pair.

	nlm	$\hat{1}$	$\hat{s}$	$\hat{t}$	$\widehat{st}$	$\widehat{ixy}$	$\widehat{ixys}$	$\widehat{ixyt}$	$\widehat{ixyst}$
$\bar{\nu}_L$	$---$	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$
e_R	$+--$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$\bar{\nu}_R$	$--+$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$
e_L	$+-+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$
$\bar{e}_R$	$-+-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$
ν_L	$++-$	$+$	$-$	$-$	$+$	$-$	$+$	$+$	$-$
$\bar{e}_L$	$-++$	$+$	$-$	$+$	$-$	$-$	$+$	$-$	$+$
ν_R	$+++$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$

the degrees of freedom that correspond to the notational product $\widehat{it}$ may also be identical.

The form of the binon potential, Eq. (31) suggests that it is possible to model the binding force as an integral of GA values over the manifold. The absence of mixing outside of the $\hat{1}$ and $\widehat{it}$ compatible pairs of binons implies that such mixtures would be too high in energy to be observed. This suggests that the $\hat{\imath}$ and $\hat{t}$ parts of the geometry have less energy associated with them, as they are of no concern in determining which binons can combine to produce low lying bound states.

APPENDIX: FOURIER SERIES AND DIRAC EQUATION

While the Dirac equation will be discussed at length in a separate paper, [4] this is a good place to derive solutions to the Dirac equation within the binon ideals shown in Fig. (5). The simplicity of the derivation of the Dirac equation, speaks well for applicability of the GA. In addition, the presence of the Dirac equation shows that the PTG, despite having no exact Poincaré symmetry, nevertheless supports a fully relativistic wave equation, and the simplicity of the Fourier series expansion suggests that the fermion families are best modeled this way.

The simplest linear differential equation in the PTG manifold, using the GA definition of the derivative, is $\partial_t \Psi = \nabla \Psi$. Written out explicitly into coordinates the equation is:

$$\partial_t \Psi(x, y, z, s; t) = (\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z + \hat{s}\partial_s)\Psi. \quad \text{(A.1)}$$

At this point, in order to make contact with the rest of this paper, we will introduce the same notational vector $\hat{t}$ that was used to distinguish position and momentum coordinates and replace the above equation with the slightly less simple one:

$$\hat{t}\partial_t \Psi(x, y, z, s; t) = (\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z + \hat{s}\partial_s)\Psi. \quad \text{(A.2)}$$

In this version, the rate of change of the position coordinates depends on the spatial derivatives of the coordinate positions and vice-versa. If we had failed to make this substitution, we would end up with half the number of copies of the Dirac equation that we expect for each family. To see the effect of using Eq. (A.1) instead of Eq. (A.2), one can replace $\hat{t}$ with 1 in the following equations.

Take a Fourier series to eliminate the s dependence, and thereby convert the Ψ a function defined on the PTG manifold to a set of functions defined on $(x, y, z; t)$, gives (for the nth fermion family):

$$\psi_n(x, y, z; t) = \int_0^{2\pi R_s} e^{ins/R_s} \Psi(x, y, z, s; t)\, ds, \quad \text{(A.3)}$$

where R_s is the radius of the hidden dimension s, and $m_n = n/R_s$ will be an effective mass. Using Eq. (A.2) to derive a differential equation for ψ_n (and multiplying on the left by $\widehat{st}$) gives:

$$\widehat{st}\partial_t \psi_n = -(\widehat{xs}\partial_x + \widehat{ys}\partial_y + \widehat{zs}\partial_z + m_n\hat{\imath}\,)\psi_n, \quad \text{(A.4)}$$

This is in the same form as the Dirac equation with the equivalences of $\gamma^0 = \widehat{st}$, $\gamma^j = \widehat{x_j s}$. Note that the 4−vector of GA constants $(\widehat{st}, \widehat{xs}, \widehat{ys}, \widehat{zs}\,)$ satisfies the same anticommutation relations as the Dirac equation gamma matrices. This shows that Eq. (A.4) is closely related to the Dirac equation. But since ψ_n takes its values from the field of GA elements, which has far more degrees of freedom than the Dirac equation's 4−vector of complex numbers, it should be clear that Eq. (A.4) is a multiple representation of the Dirac equation.

The field of GA elements in the PTG has $2^6 = 64$ real degrees of freedom, which is enough for eight copies of the Dirac equation. We can explicitly write these eight Dirac equations in geometric form by multiplying Eq. (A.4) on the right by the eight ideals $\mathcal{I}_{nlm}$ shown in Fig. (5):

$$\widehat{st}\partial_t \psi_n \mathcal{I}_{nlm} = -(\widehat{xs}\partial_x + \widehat{ys}\partial_y + \widehat{zs}\partial_z + m_n\hat{\imath}\,)\psi_n \mathcal{I}_{nlm}. \quad \text{(A.5)}$$

The result is eight copies of the Dirac equation, one for each particle.

It is also instructive to derive the general plane wave solutions to Eq. (A.4). We assume that ψ_n is in the form $\psi_n = \exp(\hat{\imath}(\mathbf{kx} - \omega t))\psi_0$ where $\mathbf{k}$ is a 3−dimensional wave vector, $\mathbf{x}$ is a 3−dimensional position, ω is a frequency, and ψ_0 is a GA constant. Specializing, as before, for particles propagating in the $+z$ direction so that $\mathbf{kx} = k_z z$, and multiplying by $\hat{\imath}$ one obtains:

$$(\omega\hat{s} - k_z\widehat{zs} - m_n)\psi_0 = 0. \quad \text{(A.6)}$$

A solution to the above is $\psi_0 = \omega\hat{s} - k_z\widehat{zs} + m_n$, subject to the condition that

$$\omega^2 - \mathbf{k}^2 = m_n^2. \quad \text{(A.7)}$$

For m_n very small, the above equation shows that $|\mathbf{k}|$ is slightly less than ω, and therefore that the particles are moving at nearly the speed of light, as would be expected of chiral fermions. The ψ_0 solution is nonzero when multiplied by any of the binon ideals listed in Fig. (5), so a solution to the Dirac equation that is in the ideal for the $|nlm>$ binon can be calculated as:

$$\psi_{nlm}(\mathbf{x};t) = e^{i(\mathbf{kx})-\omega t}\psi_0\mathcal{I}_{nlm}. \quad \text{(A.8)}$$

A general planar solution to the Dirac equation within the $\mathcal{I}_{nlm}$ ideal can be written as:

$$\psi_{nlm}(\mathbf{x};t) = e^{i(\mathbf{kx})-\omega t}\psi_0\ \kappa\, \mathcal{I}_{nlm} \quad \text{(A.9)}$$

where κ is any nonzero element of the subalgebra generated by $\{\hat{x}, \hat{y}, \hat{z}\}$.

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[10] Chiral particles in the standard model move at speed c, which is a bit of a paradox. In the PTG geometry their speed is slightly less than c due to their travel in the s dimension, and depends on which family they are in. See the appendix for the general solution to the Dirac equation.