Particle Symmetry Breaking
in
Density Matrix Formalism
using
Hestenes' Geometric Algebra

Author: Carl Brannen
Email: carl@brannenworks.com
Affiliation: Liquafaction, Woodinville, WA
Home page: www.brannenworks.com
Particle and Nuclei International Conference (PANIC05)
October 24, 2005
Santa Fe, New Mexico

Lasenby, Doran and Gull[1] have developed a gauge theory of gravity that can be formulated on flat spacetime. The formalism automatically includes spinors so the Dirac equation is incorporated in a geometrically natural way.[2] This poster provides a mechanism for breaking P and T symmetry (and therefore also C symmetry) and provides a derivation of a gauge principle for gauge bosons.

The hope of this research is to unify quantum mechanics and gravitation. In order to do this, we must match the excellent results of the standard model. This poster proposes to accomplish this by modifying the relationships between the tangent vectors of the spacetime manifold and the Clifford algebra. In a Geometric Algebra, spinors are naturally "square"[3]. One obtains a spinor by multiplying the square spinor on the right with an idempotent. Square spinors allow distinct spin$-1/2$ fermions (i.e. electron and neutrino, etc.) to be combined into a single Dirac equation. This allows a natural unification of particle types but the results will be highly symmetric. In the standard model, the distinguishing differences between different spin$-1/2$ fermions arises in their vertices which are related to probabilities and therefore squared magnitudes.

[1] A. Lasenby, C. Doran, & S. Gull, "Gravity, gauge theories and geometric algebra," Phil. Trans. R. Lond. A 356: 487-582 (1998).
[2] D. Hestenes, "Spacetime Geometry with Geometric Calculus"
[3] P. Lounesto, *Clifford Algebras and Spinors*, (Cambridge University Press, 1997)

In any physically consistent flat space theory of gravitation, the speed of light must be allowed to differ depending on position. If we allow the speed of light, c in the Dirac equation to become a Clifford algebra element c_α rather than a scalar, we can arrange for the resulting single particle (spinor) Dirac equations to be unchanged, but for the squared magnitudes, and therefore vertices, of the particles to be distinguished.

Thus the hope of following the example of condensed matter physics and arranging for the asymmetry of the elementary excitations to be determined by the asymmetry of spacetime itself. The range of values for c_α is quite limited and we parameterize the solution set for various signatures and number of spatial dimensions. We provide a method to allow the conversion of solutions back and forth between the symmetric and asymmetric generalized Dirac equations.

In addition to allowing c_α to be globally modified, we can also assume a local modification that will be differentiated. Our method of conversion between symmetric and asymmetric Dirac equations will then become a symmetry operation on the idempotents that represent the particles. Thus we can derive a gauge principle similar to that of the standard model.

A multiparticle Dirac equation from $\nabla\Psi = 0$?

In a Geometric Algebra, the gradient operator:

$$\nabla = \hat{t}\partial_t + \hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z. \tag{1}$$

is a representation of the Dirac equation in that the arithmetic rules for $\hat{x}, \hat{y}$, etc., are identical to the rules for of γ^1, γ^2... But it is a multiple representation in that you can get back the usual Dirac operator by right multiplying $\nabla\Psi$ with any one of a number of projection operators.

In the language of the Dirac matrices, this amounts to noting that the Dirac equation consists of a 4×4 matrix operator multiplying a 4×1 spinor:

$$(\gamma^\mu \partial_\mu)_{4\times 4}\, \psi_{4\times 1} = 0. \tag{2}$$

If we have four such equations, for example, one having to do with the electron, ψ_e, and the other three having to do with the three colors of the up quark, ψ_r, ψ_g, ψ_b, we can combine the four equations into one by assembling a single state matrix, Ψ out of the four spinors:

$$(\gamma^\mu \partial_\mu)_{4\times 4}\, \Psi_{4\times 4} = 0 \qquad \text{where} \tag{3}$$

$$\Psi_{4\times 4} = (\psi_e, \psi_r, \psi_g, \psi_b)\,. \tag{4}$$

To get a single particle Dirac equation from the multiparticle one, simply multiply on the right by the appropriate projection operator (in this case, a 4x4 matrix with a single one on the diagonal).

Why Density Matrices?

Given a spinor wave state $|A\rangle$, its associated density matrix is given by

$$\rho_A = |A\rangle\langle A| \tag{5}$$

The density matrix contains all the physically relevant information contained in the original spinor. But the arbitrary complex phase possessed by $|A\rangle$ is canceled out in Eq. (5). This suggests that the density matrix is the more physically relevant object.

Many physically relevant situations can be represented by density matrices that cannot be represented at all by single spinors. For example, the entropy of a mixed state can be given in simple form as a function of the density matrix:

$$S = -k\,\mathrm{tr}(\rho\ln(\rho)), \tag{6}$$

where k is Boltzmann's constant.

The rules for multiplying and adding density matrices are compatible with the rules for Clifford algebras.

What is the Schwinger Measurement Algebra?

Julian Schwinger defined a non-relativistic QFT formalism for systems with a finite number of dynamical variables. We use a gauge principle to promote this formalism to a full relativistic one that allows particle creation.

The primitive elements of the Measurement Algebra are filters that pass only a specific particle type much as a Stern-Gerlach filter may pass only a given particle spin. In the algebra, multiplication of elements corresponds to two consecutive measurements. Addition implies a less selective measurement. For example, let $M(e)$ designate a filter that allows only electrons to pass, and $M(\nu_R)$ a filter that allows only right handed neutrinos to pass. Then the sum

$$M(e + \nu_R) = M(e) + M(\nu_R) \tag{7}$$

corresponds to a measurement that passes electrons *or* right handed neutrinos. The product:

$$0 = M(e)M(\nu_R) \tag{8}$$

is zero since no particles can pass both filters.

The rules of the Schwinger measurement algebra can also be translated into a Clifford algebra. The various particle types are defined by the idempotent structure or "spectral decomposition" of the Clifford algebra.

What is the Vacuum?

In addition to measurements like $M(e)$, Schwinger's measurement algebra includes measurements that take a particle in of one type and spit it out as another type. They're written $M(a,b)$ so $M(e) = M(e,e)$. Schwinger's words[4] (slight change in symbols):

"Indeed, we can even invent a non-physical state to serve as the intermediary. We shall call this mental construct the null state 0, and write

(2.1) $\quad M(a',b') = M(a',0)M(0,b')$.

The measurement that selects a system in the state b' and produces it in the null state,

(2.2) $\quad M(0,b') = \psi(b')$,

can be described as the annihilation of a system in the state b'; and the production of a system in the state a' following its selection from the null state,

M(2.3) $\quad (a',0) = \psi^\dagger(a')$,

can be characterized as the creation of a system in the state a'. Thus the content of (2.1) is the indiscernibility of $M(a',b')$ from the compound process of the annihilation of a system in the state b' followed by the creation of a system in the state a',

(2.4) $\quad M(a',b') = \psi^\dagger(a')\psi(b')$."

[4] J. Schwinger, *Quantum Kinematics and Dynamics*, (Perseus Publishing, 1991)

What is the Geometric Algebra?

One begins with a spacetime manifold with metric signature. At each point in the manifold, one defines the tangent vectors by differentiation and chooses a basis set for the tangent vectors. For simplicity, we will work with flat spacetime and use as our basis set of tangent vectors:

$$\partial_t, \quad \partial_x, \quad \partial_y, \quad \partial_z. \tag{9}$$

With each tangent vector,we associate an element of an algebra (think of a matrix). These are called "canonical basis vectors":

$$\hat{t}, \quad \hat{x}, \quad \hat{y}, \quad \hat{z} \tag{10}$$

The squares of the canonical basis vectors are defined according to the signature (just like the γ^μ matrices):

$$\begin{array}{llll} \hat{t}^2 &= -1 \qquad & \hat{x}^2 &= +1 \\ \hat{y}^2 &= +1 \qquad & \hat{z}^2 &= +1 \end{array} \tag{11}$$

All the elements anticommute (just like the γ^μ matrices).

The algebra consists of scalar multiples along with products (as defined above) and sums (as in any vector space). There are $2^4 = 16$ degrees of freedom in this example (like the 4×4 γ^μ matrices).

Where do imaginary numbers come from in QM?

Short answer: Let σ_x, σ_y and σ_z represent the operators that measure spin$-1/2$ in their respective directions. Then their product is:

$$\sigma_x \, \sigma_y \, \sigma_z = i. \tag{12}$$

If $\vec{\sigma}$ is defined geometrically, then so is i.

Long answer: In the Schwinger measurement algebra, any given particle can be represented by the measurement that picks it out. For example

$$M(y) = (1 + \sigma_y)/2 \tag{13}$$

is the projection operator for spin$+1/2$ in the y direction.

Consider the effect of sending a particle through four Stern-Gerlach filters oriented in the z, x, y and z directions. The resulting operator is:

$$\begin{aligned} M(zxyz) &= \tfrac{1}{2}(1+\sigma_z)\tfrac{1}{2}(1+\sigma_y)\tfrac{1}{2}(1+\sigma_x)\tfrac{1}{2}(1+\sigma_z) \\ &= \tfrac{1+i}{4}(1+\sigma_z) = \tfrac{1+i}{2}M(z). \end{aligned} \tag{14}$$

Thus imaginary numbers arise in QM unconnected with wave propagation in a situation that is physically realistic and does not include an unphysical global phase invariance.

Magnitudes and Probabilities

Given that a Geometric Algebra is a vector space, the natural norm $|\ \ |^2$ defined on it is a simple sum of squares of the magnitudes of the real (or complex) coefficients. For example:

$$|1 + 3\hat{x} - 2\widehat{yzt}|^2 = 1^2 + 3^2 + 2^2 = 14. \tag{15}$$

The squared magnitude of a spin projection operator is

$$|(1 + \sigma_z)/2|^2 = 1/4 + 1/4 = 1/2. \tag{16}$$

That's because a Stern-Gerlach filter allows just half of all particles to pass. A filter that is a Schwinger "elementary selective measurement" for elementary spin$-1/2$ particles therefore has a squared magnitude of $1/2N$ where N is the number of particles.

Pure density matrices, and the elementary measurements of the Schwinger measurement algebra are both idempotents:

$$\rho^2 = \rho, \qquad M(e)^2 = M(e). \tag{17}$$

With massive neutrinos, the elementary fermions (in each generation) come in powers of two. This is good because the primitive idempotents of Clifford algebras always have squared magnitudes of 2^{-n}.

Matching the Standard Model

A particle unification shouldn't give predictions any worse than the already excellent standard model. To match those results, we need to match propagators and vertices. The (fermion) propagators are determined by the Dirac equation. The vertices are related to coupling constants and therefore probabilities.

Clifford algebras are very symmetric. The elementary particles not nearly so. We have to break the natural symmetry of the Clifford algebra, but at the same time, we can't ruin the Dirac equation.

We will modify the relationship between the tangent vectors and canonical basis vectors in such a way that the asymmetric Dirac equation still squares to the Klein-Gordon equation:

$$(\hat{t}\partial_t + \hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z)^2 = \Box \tag{18}$$

Solving the above equation gives a continuous space of modifications to the Geometric Algebra. We will call each of these a Particle Internal Symmetry Algebra (PISA).

The Particle Internal Symmetry Algebra

The solutions to the problem of squaring to the Klein Gordon equation are"asymmetric Dirac" operators of the form:

$$\nabla_\alpha = \hat{t}\partial_t + c_\alpha(\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z) \tag{19}$$

where c_α is a constant. The number of degrees of freedom for c_α depends on the number of spatial dimensions. The parameterization is:

$$c_\alpha(\alpha_t, \alpha_p) = \begin{cases} \exp(\alpha_t\hat{t} + \alpha_p\hat{p}) & \text{if } M \text{ is even,} \\ \exp(\alpha_t\hat{t}\ ) & \text{otherwise,} \end{cases} \tag{20}$$

where $\hat{p}$ is the product of the M spatial vectors.

If we possess a solution, Ψ to the usual Symmetric Dirac equation, we can convert it to a solution to the Asymmetric Dirac equation Eq. (19) by:

$$\psi_A = c_\alpha^{0.5}\psi_S c_\alpha^{-0.5}. \tag{21}$$

This substitution will even work if we're considering nonlinear differential equations as is natural in Clifford algebras.

Asymmetry in Probabilities

The transformation from symmetric to asymmetric Dirac equation solutions, Eq. (21) preserves all multiplication and addition relations on the Geometric Algebra, so all our eigenvector and eigenvalue relations will remain unchanged. But it modifies the magnitudes. For example, $c_\alpha = \cos(\delta) + \sin(\delta)\hat{t}$, with signature $\hat{t}\hat{t} = -1$, then:

$$\begin{array}{lll} \text{GA} & |\text{GA}|^2 & |\text{PISA}|^2 \\ \hat{t} & 1 & 1 \\ \hat{x} & 1 & 1 \\ \widehat{xt} & 1 & 1 \\ \hat{t}+\hat{x} & 2 & 2 \\ \hat{t}-\hat{x} & 2 & 2 \\ \hat{x}+\widehat{xt} & 2 & 2-2\sin(\delta) \\ \hat{x}-\widehat{xt} & 2 & 2+2\sin(\delta) \end{array} \quad (22)$$

Free Plane Wave Solutions

From the Schwinger measurement algebra, and an assumption of as few hidden dimensions as possible, one can show that in the context of density matrices, with particle identities defined by helicity, for the electron generation of leptons, a (pure) plane wave solution to the Dirac equation traveling in the z direction can be factored as follows:

$$\begin{aligned}\Psi &= \exp(i(z - t + \phi_0))\ \eta_z\ \iota_\chi\ \eta_z,\\ \iota_\chi^2 &= \iota_\chi\\ \eta_z &= \hat{z} + \hat{t},\\ i &= \sigma_x\ \sigma_y\ \sigma_z.\end{aligned} \tag{23}$$

In the above, ϕ_0 is the complex phase that arranges for interference depending on path length (and for interference based on Stern-Gerlach filters shown above); η_z is a nilpotent that annihilates the gradient of the exponential (and corresponds to the Grassmann algebra of the spinor formulations); ι_χ is an idempotent that corresponds to the particle type.

There are a total of 32 degrees of freedom in the factorization. Two (multiplicative) degrees are in ϕ_0, that is, $\{1, i\}$. Four degrees of freedom are in η_z, $\{1, \hat{x}, \hat{y}, \hat{z}\}$, and there are four degrees of freedom in the idempotent, $\{e_R, e_L, \nu_R, \nu_L\}$.

Covariant Derivatives and Gauges

Even though the density matrix formalism eliminates the unphysical global phase, it still allows local noncommutative gauge (or phase) dependency. Since it's all in the same algebra, the mathematics is considerably simpler than the irreducible representations of the standard model.

Let G be any element of the Clifford algebra that possesses an inverse, G^{-1}. Then if Ψ is a solution to the symmetric Dirac equation, then

$$\Psi_G = G\ \Psi\ G^{-1} \tag{24}$$

is a solution to the modified Dirac equation:

$$(G\ \nabla\ G^{-1}\ )\ \Psi_G = \nabla_G\ \Psi_G = 0. \tag{25}$$

The Clifford algebraic coefficients of ∇_G satisfy the usual rules for the canonical basis vectors of a Clifford algebra. For example,

$$(G\ \hat{t}\ G^{-1}\ )^2 = \hat{t}^2. \tag{26}$$

So a physicist unsure of what the tangent vectors are supposed to be will see ∇_G as being a perfectly good substitute for ∇.

The nilpotent and idempotent portions of Ψ will still be nilpotent and idempotent in Ψ_G. Thus G induces a gauge symmetry on the elementary particles by $\iota_\chi \rightarrow G\iota_\chi G^{-1}$.

Sample Calculation

With the caveat that the quantum numbers of quarks and leptons suggest that they are composite particles[5] so that the methods of this paper will not apply directly to them, we include a partial calculation for symmetry breaking in $1+4$ spacetime with signature $(-++++)$.

We will use $\hat{s}$ for the one hidden spatial dimension and $c_\alpha = \exp(\alpha_t\hat{t} + \alpha_p\hat{p})$, where $\hat{p} = \widehat{xyzs}$. There are a total of $2^5 = 32$ degrees of freedom in the Clifford algebra. Of these, 24 are directional, for example, $\hat{x}$, $\widehat{yz}$, $\widehat{zt}$, etc. We will restrict our attention to the 8 nondirectional degrees of freedom:

$$\begin{array}{cccc} 1 & \hat{t} & \hat{p} & \widehat{pt} \\ \hat{s} & \widehat{st} & \widehat{ps} & \widehat{pst}. \end{array} \tag{27}$$

Of these, the top four commute with $c_\alpha^{0.5}$ while the bottom convert $c_\alpha^{0.5}$ into $c_\alpha^{-0.5}$ and are therefore multiplied on the right by c_α. The overall effect of Eq. (21) on these eight is:

$$\begin{pmatrix} 1 \\ \hat{t} \\ \hat{p} \\ \widehat{pt} \\ \hat{s} \\ \widehat{st} \\ \widehat{ps} \\ \widehat{pst} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -s_pc_t & -s_ps_t \\ 0 & 1 & 0 & 0 & 0 & 0 & +s_ps_t & -s_pc_t \\ 0 & 0 & 1 & 0 & +s_pc_t & +s_ps_t & 0 & 0 \\ 0 & 0 & 0 & 1 & -s_ps_t & +s_pc_t & 0 & 0 \\ 0 & 0 & 0 & 0 & +c_pc_t & +c_ps_t & 0 & 0 \\ 0 & 0 & 0 & 0 & -c_ps_t & +c_pc_t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & +c_pc_t & +c_ps_t \\ 0 & 0 & 0 & 0 & 0 & 0 & -c_ps_t & +c_pc_t \end{pmatrix} \begin{pmatrix} 1 \\ \hat{t} \\ \hat{p} \\ \widehat{pt} \\ \hat{s} \\ \widehat{st} \\ \widehat{ps} \\ \widehat{pst} \end{pmatrix}, \tag{28}$$

where $c_t = \cos(\alpha_t)$, $s_t = \sin(\alpha_t)$, $c_p = \cosh(\alpha_p)$ and $s_p = \sinh(\alpha_p)$.

In addition to breaking the P and T symmetries, arbitrary values of α_p and α_t produce c_αs that can be used to generate global symmetries that can be gauged. Such a gauge transformation will also modify the eight nondirectional degrees of freedom as shown above. We can put the matrix into diagonal form to determine the charges associated with the symmetry:

$$\left(\ 1,\ \ 1,\ \ 1,\ \ 1,\ \ c_pe^{i\alpha_t},\ \ c_pe^{i\alpha_t},\ \ c_pe^{-i\alpha_t},\ \ c_pe^{-i\alpha_t}.\ \right) \tag{29}$$

The first four diagonal elements have charge 0. Other than the factor of c_p, the other four elements are recognizable as two particles of charge $+\alpha_t$ and two of charge $-\alpha_t$.

The c_p appearing as a factor in the diagonalized matrix is an indication that the charged particles have a different normalization than the uncharged ones. Standard quantum mechanics, where the particles are not unified, uses normalization for calculational purposes. In this theory, with the particles in a unified structure, the particle normalization must be proportional to the stress in spacetime, and thus, according to relativity, to the energy or mass. So the presence of c_p indicates that the masses of the charged particles are higher than the masses of the neutral ones.

This is exactly the form seen in the electron family leptons. That is, light neutrinos and heavy electron/positron. And the hyperbolic cosine implies a large mass hierarchy for even modest values of α_p.

In short, α_t broke the charge symmetry while α_p broke the mass symmetry in a manner similar to the leptons.

[5] C. Brannen, "The Geometry of Fermions", (2004)