The Lepton Masses

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In the standard model, the lepton masses appear as arbitrary constants determined by experiment. But in 1982, Yoshio Koide proposed a formula for the charged lepton masses that is still going strong a quarter century later. The success of Koide's formula remains unexplained, but its perfect accuracy, and its simplicity in explaining the charged lepton mass hierarchy, suggest that it may be the basis for a new theory of mass, a theory simpler than that of the standard model. In this paper, we extend the Koide mass formula to an eigenvector equation, find further coincidences, apply the formula to the neutrinos, and speculatively suggest a complete solution to the problem of the hierarchy of lepton masses and the MNS mixing matrix.

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This paper is an attempt to find and explain coincidences in the masses and mixing angles of the leptons. It is an extension of the Koide [1, 2] formula.

The paper is divided into four sections. In the first section we rewrite the Koide formula, which predicts the tau mass, as an eigenvector equation. In this form, Koide's single coincidence becomes a double coincidence, which adds to the importance and mystery of the formula.

Neutrino oscillation measurements have provided two restrictions on the masses of the three neutrinos. Koide's formula provides a third, but in its original form it is incompatible with the oscillation data. Putting the formula into eigenvector form allows it to be applied, and in the second section we derive bounds on the neutrino masses according to this restriction. Our predictions follow the usual fermion hierarchy.

The first two sections addressed the individual hierarchies of the charged and neutral leptons. In order to address the vast difference between the masses of the charged and neutral leptons, we must first define a model for mass. In the third section, we introduce a toy model based on 3×3 circulant matrices of complex numbers.

The fourth section expands the toy model of the previous section to non commutative algebra.

Defining the "mass operator" as an operator that relates the left and right handed states through a mass interaction, the fifth section proposes a model for the mass hierarchy where the neutrino masses are suppressed by a phase that appears in the mass operator of the neutrino preons. We show that the mixing angle data and the differences in the parameters of the two sectors are consistent with this assumption.

In the sixth section, we use the mass model of the previous section to predict neutrino mass values, as well as the squared mass differences for the oscillation data, to high precision.

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I. CHARGED LEPTONS

A *circulant* matrix is one of the form:

$$\Gamma(A,B,C) = \begin{pmatrix} A & B & C \\ C & A & B \\ B & C & A \end{pmatrix}. \quad (1)$$

where A, B and C are complex constants. Other authors have explored these sorts of matrices in the context of neutrino masses and mixing angles including [3–5]. Such matrices have eigenvectors of the form:

$$|n\rangle = \begin{pmatrix} 1 \\ \alpha^{+n} \\ \alpha^{-n} \end{pmatrix}, \quad n = 1,2,3. \quad (2)$$

In the above, $\alpha = e^{2i\pi/3}$. This set of eigenvectors are common to more than circulant matrices. Some references using these sets of eigenvectors in the problem of neutrino mixing include [6–10].

If we require that the eigenvalues be real, we obtain that A must be real, and that B and C are complex conjugates. This reduces the 6 real degrees of freedom present in the 3 complex constants A, B and C to just 3 real degrees of freedom, the same as the number of eigenvalues for the operators. In order to parameterize these sorts of operators, in a manner only slightly different from that chosen in [4], let us write:

$$\Gamma(\mu,\eta,\delta) = \mu \begin{pmatrix} 1 & \eta\exp(+i\delta) & \eta\exp(-i\delta) \\ \eta\exp(-i\delta) & 1 & \eta\exp(+i\delta) \\ \eta\exp(+i\delta) & \eta\exp(-i\delta) & 1 \end{pmatrix}, \quad (3)$$

where we can assume η to be non negative. While η and δ are pure numbers, μ scales with the eigenvalues. Then the three eigenvalues are given by:

$$\begin{aligned} \Gamma(\mu,\eta,\delta)\,|n\rangle &= \lambda_n\,|n\rangle, \\ &= \mu(1 + 2\eta\cos(\delta + 2n\pi/3))\,|n\rangle. \end{aligned} \quad (4)$$

The sum of the eigenvalues are given by the trace of Γ:

$$\lambda_1 + \lambda_2 + \lambda_3 = 3\mu, \quad (5)$$

and this allows us to calculate μ from a set of eigenvalues. The sum of the squares of the eigenvalues are given by the trace of Γ^2:

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 3\mu^2(1 + 2\eta^2), \quad (6)$$

and this gives a formula for η^2 in terms of the eigenvalues:

$$\frac{(\lambda_1 + \lambda_2 + \lambda_3)^2}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} = \frac{3}{1 + 2\eta^2}. \quad (7)$$

The value of δ is then easy to calculate from Eq. (4).

Putting $\eta^2 = 1/2$ in Eq. (7) gives:

$$\frac{(\lambda_1 + \lambda_2 + \lambda_3)^2}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} = \frac{3}{2}, \quad (8)$$

and this is identical in form to the formula that Yoshio Koide[1, 2] proposed in 1982 for the masses of the charged leptons:

$$\frac{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2}{m_e + m_\mu + m_\tau} = \frac{3}{2}, \quad (9)$$

provided that one associates λ_n with the square roots of the masses of the charged leptons.

Since the masses of the electron and muon are known to much greater accuracy than the mass of the tau, the Koide relation provides a prediction for the tau mass. Using the latest PDG[11] data in Mev and AMU for the electron and muon masses:

$$\begin{aligned} m_e &= 0.510998918(44) \text{ MeV} \\ &= 548.57990945(24) \text{ AMU} \end{aligned} \quad (10)$$

$$\begin{aligned} m_\mu &= 105.6583692(94) \text{ MeV} \\ &= 0.1134289264(30) \text{ AMU} \end{aligned} \quad (11)$$

the Koide relation gives an estimate for the tau mass:

$$\begin{aligned} m_\tau &= 1776.968921(158) \text{ MeV} \\ &= 1.907654627(46) \text{ AMU}, \end{aligned} \quad (12)$$

that, remarkably for a prediction of a quarter century ago, is centered in the error bars for the measured tau mass, which the PDG includes only in MeV:

$$m_\tau = 1776.99(+29 - 26) \text{ MeV}. \quad (13)$$

The error bars in the prediction Eq. (12), and in later calculations in this paper, come from assuming that the electron and muon masses are anywhere inside the error bars given by the PDG data.

If we use the PDG (MeV) data for the electron, muon and tau to find the operator that gives the square roots of the charged lepton masses, we find that μ, η^2, and δ are:

$$\begin{aligned} \mu_1 &= 17716.13(109) \text{ eV}^{0.5}, \\ \eta_1^2 &= 0.500003(23), \\ \delta_1 &= 0.2222220(19), \end{aligned} \quad (14)$$

where the subscript 1 has been added to distinguish these numbers from the figures for the neutral leptons which we will be discussing in following sections. In addition to the Koide relation which gives $\eta_1^2 = 0.5$, a new coincidence is that δ_1 is close to 2/9, a fact that went unnoticed until this author discovered it in 2005.[12]

If one supposes that $\eta_1^2 = 1/2$ and $\delta_1 = 2/9$, then one can compute the value of μ_1 individually with the electron, muon and tau. But the values one gets from the electron and muon masses are slightly incompatible. Using the MeV data, the three values one obtains (in units of square root MeV) are:

$$\begin{aligned} \mu_1 &= 17.71620000(140000) && \text{tau} \\ &= 17.71598503(79) && \text{muon} \\ &= 17.71607210(76) && \text{electron.} \end{aligned} \quad (15)$$

The discrepancy is sharper in the AMU data:

$$\begin{aligned} \mu_1 &= 0.58046396470(770) && \text{muon} \\ &= 0.58046681720(13) && \text{electron,} \end{aligned} \quad (16)$$

with units of square root AMU.

The near compatibility of the charged lepton masses with respect to $\eta^2 = 1/2$ and $\delta_1 = 2/9$ is difficult to explain. The values seem too close to be accidental. Perhaps these are the first order values and there are second order corrections that modify μ_1, η_1^2, and or δ_1.

If we assume that the Koide relation is perfect, we can use the PDG data to compute the possible values for μ_1 and δ_1. The electron and muon data are by far the most accurate, so for a given set of electron and muon masses, we compute the tau mass from the Koide relation, and then compute μ_1 and δ_1. Letting the electron and muon masses range over the PDG values in AMU and MeV, the resulting μ_1 and δ_1 values are:

$$\begin{aligned} \mu_1 &= 17715.99225(79) && \text{eV}^{0.5}, \\ &= 0.5804642012(71) && \text{AMU}^{0.5}, \end{aligned} \quad (17)$$

$$\begin{aligned} \delta_1 &= 0.22222204715(312) && \text{(from MeV)}, \\ &= 0.22222204717(48) && \text{(from AMU)}, \end{aligned} \quad (18)$$

so the data are compatible with only δ_1 requiring a second order correction.

On the other hand, if we assume that $\delta_1 = 2/9$, we can compute the tau mass from the electron and muon data and then find the values of μ_1 and η_1^2 compatible with them. The results are:

$$\begin{aligned} \mu_1 &= 17715.98230(84) && \text{eV}^{0.5}, \\ &= 0.5804638752(79) && \text{AMU}^{0.5}, \end{aligned} \quad (19)$$

$$\begin{aligned} \eta_1^2 &= 0.49999978688(380) && \text{(from MeV)}, \\ &= 0.49999978689(58) && \text{(from AMU)}. \end{aligned} \quad (20)$$

and the data are compatible with only η_1^2 requiring a second order correction.

Note that δ_1 is close to a rational number, while the other terms that are added to it inside the cosine of Eq. (4) are rational multiples of pi. This distinction follows our parameterization of the eigenvalues in that the rational fraction part comes from the operator Γ, while the $2n\pi/3$ term comes from the eigenvectors. Rather than depending on the details of the operator, the $2n\pi/3$ depends only on the fact that the operator has the symmetry of a circulant matrix.

II. NEUTRINOS

We will use m_1, m_2, and m_3 to designate the masses of the neutrinos. The experimental situation with the neutrinos is primitive at the moment. The only accurate measurements are from oscillation experiments, and are for the absolute values of the differences between squares of neutrino masses. Recent 2σ data from [13] are:

$$\begin{array}{rcll} |m_2^2 - m_1^2| &=& 7.92(1 \pm .09) \times 10^{-5} & \mathrm{eV}^2 \\ |m_3^2 - m_2^2| &=& 2.4(1 + 0.21 - 0.26) \times 10^{-3} & \mathrm{eV}^2 \end{array} \qquad (21)$$

Up to this time, attempts to apply the Koide mass formula to the neutrinos have failed,[14–16] but these attempts have assumed that the square roots of the neutrino masses must all be positive.[1] Without loss of generality, we will assume that μ_0 and η_0 are both positive, thus there can be at most one square root mass that is negative, and it can be only the lowest or central mass.

Of these two cases, the one having the central mass with a negative square root is incompatible with the oscillation data. Having the lightest neutrino have a negative square root is compatible with the oscillation data. Applying the constraint that $\eta_0^2 = 1/2$ to the oscillation error bars gives the following restrictions on the neutrino masses:

$$\begin{array}{rcll} m_1 &=& 0.000388(46) & \mathrm{eV}, \\ m_2 &=& 0.00895(17) & \mathrm{eV}, \\ m_3 &=& 0.0507(30) & \mathrm{eV}. \end{array} \qquad (22)$$

These masses can satisfy the squared mass differences of Eq. (21) as well as the Koide relation as follows:

$$\frac{(-\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3})^2}{m_1 + m_2 + m_3} = \frac{3}{2}. \qquad (23)$$

The same computation gives values for the parameters of the circulant matrix for the neutrino square root masses. Using 0 as a subscript to distinguish the neutrinos from the charged leptons, we have:

$$\begin{array}{rcl} \mu_0 &=& 0.1000(26) \quad \mathrm{eV}^{0.5}, \\ \delta_0 &=& 0.486(21). \end{array} \qquad (24)$$

[1] This restriction is difficult to understand given several papers that have assumed that the neutrino masses themselves may be negative.[17, 18]

It is not a surprise that adding the freedom of negative square roots to the Koide relation allows the oscillation data to be fitted to the neutrinos. We can only hope that these numbers will prove as prescient as Koide's.

III. A COMMUTATIVE TOY MODEL

The circulant matrix paramaters μ and δ of the charged and neutral leptons appear to have little to do with one another, but there is a speculative way of interpreting them that may give insight into the origin of mass. In this section, we begin with a toy model for mass. In the following section we will expand this to a non commutative model.

Let us consider why it is that the Koide relationship gives a simple form for the square root mass operator instead of for the mass operator. Considering mass as a coupling between left and right handed states $|L\rangle$ and $|R\rangle$, we write:

$$\begin{array}{rcl} m^2 &=& \langle R|M|L\rangle\langle L|M|R\rangle \\ &=& \mathrm{tr}\,(|R\rangle\langle R|M|L\rangle\langle L|M)\,, \end{array} \qquad (25)$$

where M is an operator that gives mass as a coupling constant between left and right handed states. Suppose that we can split the M operator into bra and ket form:

$$M = |M\rangle\ \langle M|. \qquad (26)$$

Then the Koide relation becomes natural if we have that

$$\langle M|R\rangle = \langle M|L\rangle = \sqrt{m}. \qquad (27)$$

Writing the M matrix in this form suggests that we should consider the mass operator not as an operator, but instead as an intermediate state between the left and right handed states. We can analyze the problem as one of interactions between pure density matrices.

Pure density matrices satisfy an idempotency relationship:

$$\rho^2 = \rho. \qquad (28)$$

Let us find the 3×3 complex circulant matrices that are also idempotent:

$$\begin{pmatrix} A & B & C \\ C & A & B \\ B & C & A \end{pmatrix}^2 = \begin{pmatrix} A & B & C \\ C & A & B \\ B & C & A \end{pmatrix}. \qquad (29)$$

The above gives three complex equations in three unknowns that can each be arranged to be in idempotent form, $\rho^2 = \rho$ where ρ is a linear combination of A, B, and C. The solutions are each 0 or 1 independently, and we have:

$$\begin{array}{rcl} A + B + C &=& a_1, \\ A - (B+C)/2 + i\sqrt{3}(B-C)/2 &=& a_2, \\ A - (B+C)/2 - i\sqrt{3}(B-C)/2 &=& a_3, \end{array} \qquad (30)$$

where a_n is either 0 or 1. We can conveniently denote the eight resulting idempotent circulant matrices by $\rho_{a_1 a_2 a_3}$.

Of these eight idempotent circulant matrices, ρ_{000} is just the zero matrix while ρ_{111} is the unit matrix. The matrices with two 1s, ρ_{011}, ρ_{101}, and ρ_{110} can be written as sums of ρ_{001}, ρ_{010}, and ρ_{100}, so it is only these last three that are "primitive idempotents" and therefore that correspond to elementary particles.[2] It is these three matrices that we will associate with the three generations. They can be factored into bra ket form (subject to multiplication by an arbitrary phase). For example:

$$\rho_{100} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}. \tag{31}$$

The full set of factored circulant primitive idempotent 3×3 matrices are:

$$\begin{aligned} \langle 100| &= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}, \\ \langle 010| &= \begin{pmatrix} 1 & \alpha & \alpha^* \end{pmatrix}, \\ \langle 001| &= \begin{pmatrix} 1 & \alpha^* & \alpha \end{pmatrix}, \end{aligned} \tag{32}$$

where $\alpha = e^{2i\pi/3}$. With this basis, the bra form for the mass operator is:

$$\langle M| = \mu(1, \eta e^{+i\delta}, \eta e^{-i\delta}). \tag{33}$$

The resulting mass operator $|M\rangle\langle M|$ is not of a simple form, but it does separate the rational contributions to the cosine (i.e. δ) from the irrational parts (i.e. $2n\pi/3$). To simplify the formula for mass we have to consider matrices of non commuting numbers.

IV. A NON COMMUTATIVE MODEL

In this section we will be working with the simplest non commutative algebra, the Pauli algebra. Let $\vec{r}$, $\vec{g}$, and $\vec{b}$ be three 3−dimensional vectors that define an equilateral spherical triangle. That is, the angles between them are all equal:

$$\vec{r} \cdot \vec{g} = \vec{g} \cdot \vec{b} = \vec{b} \cdot \vec{r} = \cos(\theta_\angle), \tag{34}$$

where $\theta_\angle$ is the angle between the vectors. We will use these three vectors to define a subalgebra of the usual Pauli algebra.

The projection operators for spin in the directions of these three vectors are defined as:

$$\begin{aligned} R &= (1 + \vec{r} \cdot \vec{\sigma})/2, \\ G &= (1 + \vec{g} \cdot \vec{\sigma})/2, \\ B &= (1 + \vec{b} \cdot \vec{\sigma})/2, \end{aligned} \tag{35}$$

where $\vec{\sigma}$ is the usual vector of basis vectors of the Pauli algebra. For example, the projection operator for spin in the $+z$ direction is $(1 + \sigma_z)/2$.

Let us consider the subgroup of the Pauli algebra defined by complex multiples of arbitrary products of these three projection operators. A typical element of the algebra might look like:

$$\beta RGGBRRRB, \tag{36}$$

where β is a complex number. In any such product, we can collapse squares of projection operators. So the above is equal to:

$$\beta R(GG)B(RRR)B = \beta RGBRB. \tag{37}$$

Next, any product of the form UVU where U and V are two different projection operators, can be reduced to a real multiple of U. In the above example:

$$BRB = \frac{1 + \cos(\theta_\angle)}{2}\, B, \tag{38}$$

so we can reduce our example accordingly:

$$\beta RG(BRB) = \beta \frac{1 + \cos(\theta_\angle)}{2}\, RGB. \tag{39}$$

Finally, products of the form UVW where U, V and W are distinct projection operators, can be reduced to complex multiples of the product UW. For the example of our equilateral projection operators, the rule[3] is:

$$RGB = \sqrt{\frac{1 + \cos(\theta_\angle)}{2}}\, e^{+i\phi}\, RB, \tag{40}$$

where ϕ is half the (oriented) area of the spherical triangle defined by the vectors $\vec{r}$, $\vec{g}$, and $\vec{b}$. These rules allow any product of R, G, and B to be reduced to a complex multiple of the first and last operators. In addition, the complex multiple will be of the form:

$$\left(\frac{1 + \cos(\theta_\angle)}{2}\right)^{j/2} e^{ik\phi}, \tag{41}$$

where j is the number of terms reduced from the product other than by the square rule, and k counts the number of times one loops around the RGB sequence in the positive direction.

With these rules, we can now upgrade our bras and kets to non commutative form:

$$\begin{aligned} \begin{pmatrix} a \\ b \\ c \end{pmatrix} &\to \begin{pmatrix} aR \\ bG \\ cB \end{pmatrix}, \\ \begin{pmatrix} a^* & b^* & c^* \end{pmatrix} &\to \begin{pmatrix} a^*R & b^*G & c^*B \end{pmatrix}, \end{aligned} \tag{42}$$

[2] This interpretation follows Julian Schwinger. See [19] for the definition of "elementary measurements" or consider the case of the 2×2 circulant idempotent matrices which happen to be pure density matrices in the Pauli algebra.

[3] The reader unfamiliar with this relation is invited to verify it. For example, $(1+\sigma_z)(1+\sigma_y)(1+\sigma_x)/8 = \sqrt{1/2}e^{+i\pi/4}(1+\sigma_z)(1+\sigma_x)$, and the spherical area of the first quadrant is $\pi/2$.

where a, b, and c are arbitrary complex numbers. When converting a vector to density matrix form we now end up with a matrix of products of projection operators:

$$\begin{pmatrix} aR \\ bG \\ cB \end{pmatrix} \begin{pmatrix} a^*R & b^*G & c^*B \end{pmatrix} = \begin{pmatrix} a^*aR & b^*aRG & c^*aRB \\ a^*bGR & b^*bG & c^*bGB \\ a^*cBR & b^*cBG & c^*cB \end{pmatrix}. \tag{43}$$

Instead of considering matrices of complex numbers, we now have matrices of non commuting numbers.

If we have two matrices of this sort, we can use our reduction formulas to compute the matrix product. For example, define the matrices of projection operators $\hat{a}$ and $\hat{b}$:

$$\begin{aligned} \hat{a} &= \begin{pmatrix} a_{11}R & a_{12}RG & a_{13}RB \\ a_{21}GR & a_{22}G & a_{23}GB \\ a_{31}BR & a_{32}BG & a_{33}B \end{pmatrix}, \\ \hat{b} &= \begin{pmatrix} b_{11}R & b_{12}RG & b_{13}RB \\ b_{21}GR & b_{22}G & b_{23}GB \\ b_{31}BR & b_{32}BG & b_{33}B \end{pmatrix}, \end{aligned} \tag{44}$$

where a_{nm} and b_{nm} are complex constants. Then the product of the two matrices is of the same form. In computing the diagonal terms of the product, one comes upon terms of the form RGR and these reduce by Eq. (38) to give, for example:

$$(ab)_{11} = a_{11}b_{11} + \frac{1+\cos(\theta_\angle)}{2}(a_{12}b_{21} + a_{13}b_{31}). \tag{45}$$

The off diagonal terms of the product have terms of the form RBG which reduce by Eq. (40). For example, the RG term is:

$$(ab)_{12} = a_{11}b_{12} + a_{12}b_{22} + \sqrt{\frac{1+\cos(\theta_\angle)}{2}}e^{-i\phi}a_{13}b_{32}. \tag{46}$$

The sign of the exponential, as defined in Eq. (40), will be negative for the RG, GB and BR terms, and positive for the others.

In Eq. (45) and Eq. (46) we have rules that will allow us to do matrix multiplication but we can do better. Consider the transformation:

$$a'_{nm} = \begin{cases} a_{nm} & \text{if } n = m, \\ a_{nm}\sqrt{\frac{2}{1+\cos(\theta_\angle)}}e^{+i\phi/3} & \text{if } n = m+1, \\ a_{nm}\sqrt{\frac{2}{1+\cos(\theta_\angle)}}e^{-i\phi/3} & \text{if } n = m-1. \end{cases} \tag{47}$$

This transformation will take the rules of Eq. (45) and Eq. (46) to the usual matrix multiplication:

$$\begin{aligned} (a'b')_{11} &= a'_{11}b'_{11} + a'_{12}b'_{21} + a'_{13}b'_{31}, \\ (a'b')_{12} &= a'_{11}b'_{12} + a'_{12}b'_{22} + a'_{13}b'_{32}. \end{aligned} \tag{48}$$

The transformation preserves addition, that is, $(a+b)'_{nm} = a'_{nm} + b'_{nm}$, and so any problem defined over addition and multiplication of the matrices of projection operators can be converted into a problem of addition and multiplication of complex matrices.

In the previous section, we solved the problem of finding the primitive idempotents among circulant 3×3 complex matrices. In order to reduce the length of our formulas, define:

$$T = \sqrt{\frac{1+\cos(\theta_\angle)}{2}}. \tag{49}$$

Then the circulant primitive idempotents among the 3×3 matrices of projection operators are the three matrices $|n\rangle\langle n|$:

$$\frac{1}{3}\begin{pmatrix} R & \frac{e^{+i(\phi+2n\pi)/3}}{T}RG & \frac{e^{-i(\phi+2n\pi)/3}}{T}RB \\ \frac{e^{-i(\phi+2n\pi)/3}}{T}GR & G & \frac{e^{+i(\phi+2n\pi)/3}}{T}GB \\ \frac{e^{+i(\phi+2n\pi)/3}}{T}BR & \frac{e^{-i(\phi+2n\pi)/3}}{T}BG & B \end{pmatrix}. \tag{50}$$

To obtain the mass formula $\mu(1+\sqrt{2}\cos(\delta+2n\pi/3))$ from the above, we put $T = \sqrt{1/2}$, $\phi = 3\delta$ and, recalling that $\mathrm{tr}(RG) = T^2$, take the trace of the sum of elements of the matrix.[4] For example, the trace of the sum over the top row is:

$$\begin{aligned} \sqrt{m_n} &= \mathrm{tr}\left(R + \frac{e^{+i(\phi+2n\pi)/3}}{T}RG + \frac{e^{-i(\phi+2n\pi)/3}}{T}RB\right)/3, \\ &= (1 + 2T\cos(\phi/3 + 2n\pi/3))/3. \end{aligned} \tag{51}$$

The other two rows give the same. The only problem is that $\mu = 1$. If there were only one lepton sector we'd be done. To explain the lepton hierarchy, we need to slightly complicate our mass model. Fortunately, the lepton mixing matrix gives a hint on how to do this.

V. THE LEPTON MASS HIERARCHY

Let us assume the tribimaximal[20–22] lepton mixing matrix:

$$U = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{52}$$

The above matrix gives the transformations that must be performed upon the neutrino weak eigenstates $(\nu_e, \nu_\mu, \nu_\tau)^t$ to transform them into the mass eigenstates $(\nu_1, \nu_2, \nu_3)^t$. In the context of circulant matrices, it makes more sense to convert this matrix into one that converts from $(\nu_e, \nu_\mu, \nu_\tau)^t$ to (ν_R, ν_G, ν_B).

To convert Eq. (52) from mass form to circulant form, we must left multiply it by a matrix of circulant eigenvectors. There are many ways we could do this, depending

[4] It should be noted that for the Pauli algebra, setting $T = \sqrt{1/2}$ forces $\phi = \pm\pi/4$ which is incompatible with the mass data. To get $\phi = 3\delta$ at the same time as $T = \sqrt{1/2}$ we must use a more complicated algebra, for example $\mathcal{CL}(4,1)$. For such an algebra, the derivation is similar to that shown.

on the order we choose, but a choice that leads to a particularly simple form for the product is the following:

$$\frac{1}{\sqrt{3}}\begin{pmatrix}1 & \alpha & \alpha^* \\ 1 & 1 & 1 \\ 1 & \alpha^* & \alpha\end{pmatrix}\begin{pmatrix}\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}\sqrt{1/2} & 0 & -i\sqrt{1/2} \\ 0 & 1 & 0 \\ \sqrt{1/2} & 0 & i\sqrt{1/2}\end{pmatrix} \quad (53)$$

The above matrix is particularly simple in that it is a 24th root of unity. A possible explanation for this is that the mass relation of neutrinos is not as simple as it is with electrons. Such a difference can explain both the differences in both μ and δ between the sectors.

The coupling between left and right eigenstates given in Eq. (25) made the assumption that the mass interaction had no intrinsic phase. That is, if we define a mass operator M that takes left handed states to right handed states and right handed states to left handed states:

$$\begin{aligned} M|R\rangle &= |L\rangle, \\ M|L\rangle &= |R\rangle, \end{aligned} \quad (54)$$

we have assumed that $M^2 = 1$. A more general assumption is that M^2 is complex:

$$M^2|L\rangle = p^2 e^{2i\kappa}|L\rangle. \quad (55)$$

If $\kappa = 2\pi/24 = \pi/12$, then the operator that brings $|L\rangle$ back to a multiple of $|L\rangle$ is M^{24}:

$$M^{24}|L\rangle = p^{24}|L\rangle. \quad (56)$$

This brings in a factor of p^24, which for values of p close to unity is enough to explain the mass difference between the charged and neutral leptons.

More particularly, let us suppose that the M^2 operator acts on the electron preons as follows:

$$M^2|e_L\rangle = p^2|e_L\rangle, \quad (57)$$

but that the same operator, when acting on the neutrino preons picks up a phase giving:

$$M^{24}\nu_L\rangle = p^{24}|\nu_L\rangle. \quad (58)$$

Then there will be a difference in the masses of the two sectors of p^{22}.

The ratio of the masses of the charged leptons over the neutral ones is given by $(\mu_1/\mu_0)^2$. The experimental data from Eq. (17) and Eq. (24) give:

$$\begin{aligned} (\mu_1/\mu_0)^2 &= (17715.99225(79)/0.1000(26))^2 \\ &= 3.14(16) \times 10^{10} \\ &= (2.9999(71))^{22} \end{aligned} \quad (59)$$

This is consistent with the mass operator being one third of a 24th root of unity.

When the lepton mixing matrix is converted into circulant form in Eq. (53), it becomes a 24th root of unity in a specific way. The electron and tau neutrinos are rotated, but the muon is left unchanged. Examining the circulant matrices, we can suppose that it is the muon which corresponds to the $(1, 1, 1)$ solution and the electron and tau that are the solutions with complex phases. From the transformation to a non commutative algebra, this suggests that there should be a change to the δ phase between the lepton sectors. From Eq. (18) and Eq. (25) we have:

$$\begin{aligned} \delta_1 - \delta_0 &= 0.22222204717(48) - 0.486(21) \\ &= -0.264(21) \\ &= -\tfrac{\pi}{12}1.008(80) \end{aligned} \quad (60)$$

This is consistent with an angular difference between the two sectors of $\pi/12$, what one might expect from a mass interaction that requires 24 stages in one sector and only 2 in the other.

VI. PRECISION NEUTRINO MASS PREDICTIONS

With the model of the previous section, it becomes possible to predict the neutrino masses, and therefore the differences between their squares, to very high precision. We do this by using the electron and muon masses to compute δ_1 and μ_1, and then use supposing $\mu_1/\mu_0 = 3^{11}$ and $\delta_1 - \delta_0 = -\pi/12$. The resulting neutrino masses are:

$$\begin{aligned} m_1 &= 0.000383462480(38) && \text{eV} \\ &= 0.4116639106(115) \times 10^{-12} && \text{AMU} \end{aligned} \quad (61)$$

$$\begin{aligned} m_2 &= 0.00891348724(79) && \text{eV} \\ &= 9.569022271(246) \times 10^{-12} && \text{AMU} \end{aligned} \quad (62)$$

$$\begin{aligned} m_3 &= 0.0507118044(45) && \text{eV} \\ &= 54.44136198(131) \times 10^{-12} && \text{AMU} \end{aligned} \quad (63)$$

Similarly, the predictions for the differences of the squares of the neutrino masses are:

$$\begin{aligned} m_2^2 - m_1^2 &= 7.930321129(141) \times 10^{-5} && \text{eV}^2 \\ &= .913967200(47) \times 10^{-24} && \text{AMU}^2 \end{aligned} \quad (64)$$

$$\begin{aligned} m_3^2 - m_2^2 &= 2.49223685(44) \times 10^{-3} && \text{eV}^2 \\ &= 2872.295707(138) \times 10^{-24} && \text{AMU}^2 \end{aligned} \quad (65)$$

As with the Koide predictions for the tau mass, these predictions for the squared mass differences are dead in the center of the error bars. We can only hope that the future will show our calculations to be as prescient as Koide's.

That the masses of the leptons should have these sorts of relationships is particularly mysterious in the context of the standard model.[23] It is hoped that this paper will stimulate thought among theoreticians. Perhaps the fundamental fermions are bound states of deeper objects.

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[1] Y. Koide, *Fermion-Boson Two Body Model of Quarks and Leptons and Cabibbo Mixing*, Lettere al Nuovo Cimento **34**, 201 (1982).

[2] Y. Koide, *A Fermion-Boson Composite Model of quarks and leptons*, Phys Lett. B **120**, 161 (1983).

[3] S. L. Adler, *Model for Particle Masses, Flavor Mixing, and CP Violation, Based on Spontaneously Broken Discrete Chiral Symmetry as the Origin of Families*, Phys. Rev. D **59**, 015012 (1999).

[4] Y. Koide, *Quark and lepton mass matrices with a cyclic permutation invariant form* (2000), hep-ph/0005137.

[5] P. F. Harrison, D. H. Perkins, and W. G. Scott, *Threefold maximal lepton mixing and the solar and atmospheric neutrino deficits*, Phys. Lett. B **349**, 137 (1995).

[6] N. Cabibbo, *Time Reversal Violation in Neutrino Oscillation*, Phys. Lett. B **72**, 333 (1978).

[7] L. Wolfenstein, *Oscillations among three neutrino types and CP violation*, Phys. Rev. D **18**, 958 (1978).

[8] A. Acker, J. G. Learned, S. Pakvasa, and T. J. Weiler, *A single solution to the atmospheric and solar neutrino anomalies*, Phys. Lett. B pp. 149–153 (1992).

[9] R. N. Mohaptra and S. Nussinov, *Almost degenerate neutrinos with maximal mixing*, Phys. Lett. B **346**, 75 (1995).

[10] C. Giunti, C. W. Kim, and J. D. Kim, *Atmospheric neutrino problem in maximally-mixed three generations of neutrinos*, Phys. Lett. B **352**, 357 (1995).

[11] S. Eidelman, K. Hayes, K. Olive, M. Aguilar-Benitez, C. Amsler, D. Asner, K. Babu, R. Barnett, J. Beringer, P. Burchat, et al., *Review of Particle Physics, (Particle Data Group Data and 2005 partial update for edition 2006)*, Physics Letters B **592**, 1+ (2004), URL `http://pdg.lbl.gov`.

[12] A. Rivero and A. Gsponer, *The strange formula of Dr. Koide* (2005), hep-ph/0505220.

[13] G. Fogli, E. Lisi, A. Marrone, and A. Palazzo, *Global analysis of three-flavor neutrino masses and mixings* (2005), hep-ph/0506083.

[14] Nan Li and Bo-Qiang Ma, *Estimate of neutrino masses from Koide's relation*, Phys. Lett. B **609**, 309 (2005), hep-ph/0505028.

[15] J. M. Gérard, F. Goffinet, and M. Herquet, *A New Look at an Old Mass Relation*, Phys. Lett. B **633**, 563 (2006), hep-ph/0510289.

[16] R. N. Mohapatra and A. Y. Smirnov, *Neutrino mass and new physics* (2006), hep-ph/0603118.

[17] Xiao-Gang He and A. Zee, *Neutrino Masses with "Zero Sum" Condition:* $m_{\nu_1} + m_{\nu_2} + m_{\nu_3} = 0$, Phys. Rev. D **68**, 037302 (2003), hep-ph/0302201.

[18] K. Matsuda, *Phenomenological analysis of lepton and quark Yukawa couplings*, Phys. Rev. D **69**, 113006 (2004), hep-ph/0401154.

[19] J. Schwinger, *Quantum Kinematics And Dynamics* (Perseus Publishing, 1991).

[20] P. Harrison, D. H. Perkins, and W. G. Scott, Phys. Lett. B **458**, 79 (1999), hep-ph/9904297.

[21] P. F. Harrison, D. H. Perkins, and W. G. Scott, Phys. Lett. B **530**, 167 (2002), hep-ph/0202074.

[22] P. F. Harrison and W. G. Scott (2004), hep-ph/0402006.

[23] Y. Koide, *Challenge to the mystery of the charged lepton mass formula* (2005), hep-ph/0506247.