Foundations of Physics manuscript No.
(will be inserted by the editor)

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Path Integrals and the Weak Force

Received: 8/31/2009 / Accepted: date

Abstract In a previous paper, we showed that spin-1/2 can arise from a more primitive form of spin called "tripled Pauli spin", along with three generations of elementary fermions. This gave a possible explanation for various coincidences in the fermion masses and mixing matrices. In this paper we continue the analysis. We show that the weak hypercharge (t_0), and weak isospin (t_3) quantum numbers can be derived from the long term propagators of three tripled Pauli spin particles. This completes a derivation of the standard model elementary fermions.

Keywords Feynman integral · quantum information theory · spin · elementary particles · tripled Pauli spin · weak hypercharge · weak isospin

PACS 12.40.Yx · 12.38.Aw

1 Introduction

Two bases are mutually unbiased if their transition probabilities are all equal. The most common example of mutually unbiased bases are position and momentum. George Svetlichny[1] showed that the usual (position) path integrqals can be interpreted as arising from products of mutually unbiased bases. He suggested that from an ontological point of view, Feynman integrals are more fundamental than Lagrangians. If this is the case, instead of simple Lagrangians (defined by symmetry principles) we should look for simple Feynman path integrals; path integrals that define the observed symmetries as emergent properties.

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Julian Schwinger [2] took the properties of the complementary observables position and momentum and wrote them, using the discrete Fourier transform, into ν-dimensional Hilbert space for ν prime. By taking the limit as $\nu \to \infty$, Schwinger obtained the complementary observables position and momentum in one dimension. Jiří Tolar and Goce Chadzitaskos [3] showed that this sequence amounts to quantum mechanics on the lattice and indeed Svetlichny's interpretation is true in the limit.

For the simplest example of Hilbert space, spin-1/2, there are three bases that are pairwise mutually unbiased, spin in the $\pm\mathbf{x}$, $\pm\mathbf{y}$, and $\pm\mathbf{z}$ directions. With the above previous work, it's natural to consider path integrals over these six states. A particle moving along such a path can take six possible states at each vertex. This is three times the usual number for the Pauli spin matrices so we will call this "tripled Pauli spin".

To define a basis for spin-1/2 we first choose an arbitrary unit vector $+\mathbf{v}$, define it as "up" and the opposite direction $-\mathbf{v}$, as "down". On measuring spin with respect to $\mathbf{v}$, a spin-1/2 particle with be found to have spin-up $+\mathbf{v}$ or spin-down $-\mathbf{v}$.

A basis for tripled Pauli spin requires that we choose a perpendicular coordinate system for space, that is, three perpendicular unit vectors $+\mathbf{x}$, $+\mathbf{y}$, and $+\mathbf{z}$. A measurement of tripled Pauli spin will give one of the six unit vectors $\pm\mathbf{x}$, $\pm\mathbf{y}$, or $\pm\mathbf{z}$. In our "previous paper" [4], we analyzed path integrals consisting of sequences of tripled Pauli spin oriented in the $+\mathbf{x}$, $+\mathbf{y}$, and $+\mathbf{z}$ directions. We define the projection operators for spin in these directions as:

$$\begin{aligned} X &= (1+\sigma_x)/2, \\ Y &= (1+\sigma_y)/2, \\ Z &= (1+\sigma_z)/2, \end{aligned} \tag{1}$$

and call these "forms" to distinguish them from the "states" of spin-1/2. An example spin path, which passes through four forms, is shown in Fig. (1).

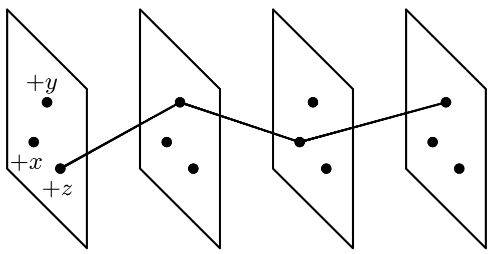

Fig. 1 Spin path $YXYZ$. This contributes to the amplitude for propagation from the state $|+z\rangle$ to $|+y\rangle$.

The mutually unbiased bases define only a finite number of states. Our previous paper showed that even though the Pauli spin matrices do not commute, restricting our attention to a set of mutually unbiased bases allows us to compute spin path integrals using the multiplication of complex numbers. In this paper we will use this result and assume commutative multiplication rules implicitly.

2 Three Particle Spin Paths

The spin path integrals analyzed in the previous paper were summations over paths for a single particle. The present paper extends the previous analysis to three particles, with the three particles $\{1, 2, 3\}$ taking the three forms X, Y, Z. We will assume the particles to be fermions so that each form takes exactly one particle.

Configuration space for three particles is a 27-dimensional space:

$$\{X, Y, Z\} \times \{X, Y, Z\} \times \{X, Y, Z\}, \tag{2}$$

however we will assume that the particles satisfy the Pauli exclusion principle. This eliminates all but six of the 27 configurations. We will label these six as follows:

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline I & X & Y & Z \\ J & Y & Z & X \\ K & Z & X & Y \\ R & X & Z & Y \\ G & Z & Y & X \\ B & Y & X & Z \end{array} \tag{3}$$

Previously our spin paths were sequences of forms taken from $\{X, Y, Z\}$ such as $YXYZ$. Now our spin paths will be sequences taken from $\{I, J, K, R, G, B\}$.

Our new spin paths give the paths taken by three particles. For example $IKGR$ defines the following sequence of configurations:

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline R & X & Z & Y \\ G & Z & Y & X \\ K & Z & X & Y \\ I & X & Y & Z \end{array} . \tag{4}$$

Particle 1 is sent through the path $XZZX$, particle 2 is sent through $YXYZ$, and particle 3 passes through $ZYXY$. See Fig. (2).

Our previous paper showed how to reduce the non commutativity of the spin path over MUBs of a single particle to a complex number. Since spin paths over MUBs for three particles naturally split into three spin paths for single particles, the same method applies to the present case. The complex number for the three particle spin path is the product of the three complex numbers for the individual spin paths.

For the spin path illustrated in Fig. (2) $IKGR$, the overall complex number for the amplitude is the (tensor) product of those derived from the three

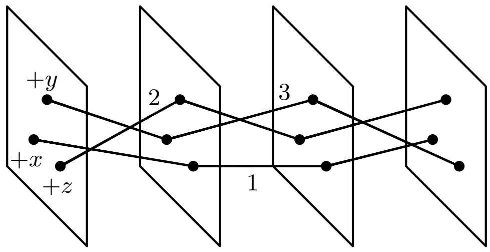

Fig. 2 Spin path $IKGR$. The three particles are labeled 1, 2, and 3. This contributes to the amplitude for propagation from the state R to I.

single particle spin paths $XZZX$, $YXYZ$, and $ZYXY$. Using the methods of the previous paper, these are computed as:

$$\begin{array}{l} XZZX = 1/2X \quad\ \to 1/2, \\ YXYZ = 1/2YZ \to \eta_g/2, \\ ZYXY = 1/2ZY \to \eta_g^*/2, \end{array} \tag{5}$$

where $\eta_g = \sqrt{1/2}e^{+i\pi/12}e^{2ig\pi/3}$ was derived in the previous paper, and $g = 1, 2, 3$ gives the particle generation. Multiplying the numbers in the right hand column of the above equation, the complex number defining the amplitude for $IKGR$ is 1/16. In terms of configuration space, the path takes the $R = X \times Z \times Y$ initial state to the $I = X \times Y \times Z$ final state.

The previous paper associated the three generations with complex phases $w^g = \exp(2ig\pi/3)$. These were applied to the nine single particle "spin path basis" in the pattern:

$$\begin{pmatrix} XX & w^{+g}XY & w^{-g}XZ, \\ w^{-g}YX & YY & w^{+g}YZ, \\ w^{+g}ZX & w^{-g}ZY & ZZ \end{pmatrix}, \tag{6}$$

that is, the XY component is replaced with $w^{+g}XY$ in generation g, etc.

The present paper works in a configuration space of $\{I, J, K, R, G, B\}$. These live in the tensor product of three copies of the spin path basis and are also transformed under generation. Computing, we find

$$\begin{array}{rcl} I = XX \times YY \times ZZ \to & XX \times YY \times ZZ & = XX \times YY \times ZZ, \\ J = XY \times YZ \times ZX \to & w^{+g+g+g}XY \times YZ \times ZX & = XY \times YZ \times ZX, \\ K = XZ \times YX \times ZY \to & w^{+g+g+g}XZ \times YX \times ZY & = XZ \times YX \times ZY, \\ R = XX \times YZ \times ZY \to & w^{+g-g}XX \times YZ \times ZY & = XX \times YZ \times ZY, \\ G = XZ \times YY \times ZX \to & w^{+g-g}XZ \times YY \times ZX & = XZ \times YY \times ZX, \\ B = XY \times YX \times ZZ \to & w^{+g-g}XY \times YX \times ZZ & = XY \times YX \times ZZ; \end{array} \tag{7}$$

where we have taken into account the fact that $w^3 = 1$. We see that I, J, K, R, G, and B do not depend on generation and so the calculations of the present paper will be generation independent.

3 Matrices

Since $XX = X$, $YY = Y$, and $ZZ = Z$, the $\{I, J, K, R, G, B\}$ are also idempotent: $II = I$, $JJ = J$, $KK = K$, etc. This means that in computing a spin product, such as $IKGR$, we can follow the lead of the previous paper and duplicate the inner entries and rearrange:

$$IKGR = IKKGGR = (IK)(KG)(GR), \tag{8}$$

so as to write any spin product in terms of the 36 objects $\{II, IJ, IK, ..., BB\}$ subject to the restriction that we will only multiply pairs whose adjacent elements match. This matches the rules for matrix multiplication so we will be considering 6×6 matrices whose entries are complex numbers associated with $\{I, J, K, R, G, B\} \times \{I, J, K, R, G, B\}$.

As in the previous paper, the product of composition of two propagators a and b is given by the matrix product ab. And as before, we will be concerned with the limit as $n \to \infty$ of G_λ^n where the matrix G_λ

$$G_\lambda = \kappa \begin{pmatrix} g_{II} & g_{IJ} & g_{IK} & g_{IR} & g_{IG} & g_{IB} \\ g_{JI} & g_{JJ} & g_{JK} & g_{JR} & g_{JG} & g_{JB} \\ g_{KI} & g_{KJ} & g_{KK} & g_{KR} & g_{KG} & g_{KB} \\ g_{RI} & g_{RJ} & g_{RK} & g_{RR} & g_{RG} & g_{RB} \\ g_{GI} & g_{GJ} & g_{GK} & g_{GR} & g_{GG} & g_{GB} \\ g_{BI} & g_{BJ} & g_{BK} & g_{BR} & g_{BG} & g_{BB} \end{pmatrix} \tag{9}$$

has entries which correspond to the amplitudes for the 36 possible propagators from $\{I, J, K, R, G, B\}$ to $\{I, J, K, R, G, B\}$. The complex number κ will be chosen to make the long term propagator G_λ^∞ preserve probability. The particle is given by λ, i.e. left handed electron, right handed top quark, etc. As before, the g_{PQ} values depend on λ.

The six entries down the diagonal of the matrix G_λ that is, g_{II}, g_{JJ}, etc., all correspond to propagators that do not permute the three particles. Topologically, these are the identity and since them must be all equal we will write them as g_I. Similarly, the six entries g_{JI}, g_{KJ}, g_{IK}, g_{GR}, g_{BG}, and g_{RB} are all the same permutation so noting that this permutation takes an initial state I to a final state J, we will abbreviate them as g_J. Continuing with this process we obtain:

$$G_\lambda = \kappa \begin{pmatrix} g_I & g_K & g_J & g_R & g_G & g_B \\ g_J & g_I & g_K & g_G & g_B & g_R \\ g_K & g_J & g_I & g_B & g_R & g_G \\ g_R & g_G & g_B & g_I & g_K & g_I \\ g_G & g_B & g_R & g_J & g_I & g_K \\ g_B & g_R & g_G & g_K & g_J & g_I \end{pmatrix}, \tag{10}$$

the complexity of the G_λ matrix is reduced from 36 complex numbers to six.

The g_R, g_G, and g_B matrix entries correspond to situations where one of the particles has the same initial and final positions, while the other two particles swap places. These are related topologically by an even permutation of the particles $\{1,2,3\}$ and consequently we will assume them all equal and write them as g_C. Similarly, g_J and g_K are converted into each other by any odd permutation of the particles and so must be related by $g_J = \pm g_K$. This reduces the number of complex numbers describing G_λ to three:

$$G_\lambda = \kappa \begin{pmatrix} g_I & g_K & g_J & g_C & g_C & g_C \\ g_J & g_I & g_K & g_C & g_C & g_C \\ g_K & g_J & g_I & g_C & g_C & g_C \\ g_C & g_C & g_C & g_I & g_K & g_I \\ g_C & g_C & g_C & g_J & g_I & g_K \\ g_C & g_C & g_C & g_K & g_J & g_I \end{pmatrix}, \tag{11}$$

subject to the additional requirement that $g_J = \pm g_K$.

4 Long Time Propagator Limit

The matrix G_λ of Eq. (11) has a form that is clearly preserved under matrix addition. What's surprising is that it is also preserved under multiplication. Thus g_I, g_J, g_K, and g_C give a 4-parameter complex subalgebra of the 6×6 matrices. Consequently, in computing

$$\lim_{n\to\infty} G_\lambda^n = G_\lambda^\infty \tag{12}$$

we will be looking for matrices of the form Eq. (11) that also are projection operators:

$$(G_\lambda^\infty)^2 = G_\lambda^\infty, \tag{13}$$

a restriction that we will solve explicitly.

Let h_I, h_J, h_K, and h_C be the four complex numbers that define G_λ^∞ in the form of Eq. (11). Then Eq. (13) gives four quadratic equations in four unknowns:

$$\begin{aligned} h_I &= h_I^2 + 2h_J h_K + 3h_C^2, \\ h_J &= 2h_I h_J + h_K^2 + 3h_C^2, \\ h_K &= 2h_I h_K + h_J^2 + 3h_C^2, \\ h_C &= 2h_C(h_I + h_J + h_K), \end{aligned} \tag{14}$$

along with $h_J = \pm h_K$. We will treat h_I, h_J, h_K, and h_C as quantum numbers that define the various elementary fermions.

Other than solutions related by $i \to -i$, there are eight solutions to Eq. (14). Four have $h_C = 0$:

h_I	h_J	h_K	h_C
0	0	0	0
1/3	+1/3	+1/3	0
2/3	−1/3	−1/3	0
1	0	0	0

(15)

We will associate the above solutions with the right handed fermions. Note that h_I is necessary to distinguish between the first and last of these solutions. The other solutions have $h_c = \pm 1/6$:

$$\begin{array}{llll} h_I & h_J & h_K & h_C \\ \hline 1/6 & 1/6 & 1/6 & +1/6 \\ 1/6 & 1/6 & 1/6 & -1/6 \\ 1/2 & i/\sqrt{12} & -i/\sqrt{12} & +1/6 \\ 1/2 & i/\sqrt{12} & -i/\sqrt{12} & -1/6 \end{array}, \tag{16}$$

which we will associate with the left handed fermions. Note that h_C is necessary to distinguish these solutions. The values of h_I and h_C are necessary and sufficient to distinguish all eight solutions. Since we are treating the h_χ as quantum numbers, (h_I, h_C) makes a complete set.

5 The Weak Quantum Numbers

Anti-particles carry quantum numbers that are the negative of the corresponding particle. This doubles the eight sets of quantum numbers given in Eq. (15) and Eq. (16) to sixteen sets of quantum numbers. This happens to be the same as the number of (handed) elementary fermions of a single generation.

The elementary fermions carry weak hypercharge t_0, and weak isospin t_3, quantum numbers. We will associate these with h_I and h_C by the relations:

$$\begin{array}{l} t_0 = 2h_I, \\ t_3 = 3h_C. \end{array} \tag{17}$$

Then the particle assignments (for the first generation) are:

	t_0 $2h_I$	t_3 $3h_C$	h_J	h_K		t_0 $2h_I$	t_3 $3h_C$	h_J	h_K
ν_L	-1	$+1/2$	$-i/\sqrt{12}$	$+i/\sqrt{12}$	$\bar{\nu}_L$	$+1$	$-1/2$	$+i/\sqrt{12}$	$-i/\sqrt{12}$
ν_R	0	0	0	0	$\bar{\nu}_R$	0	0	0	0
d_L	$+1/3$	$-1/2$	$+1/6$	$+1/6$	$\bar{d}_L$	$-1/3$	$+1/2$	$-1/6$	$-1/6$
d_R	$-2/3$	0	$-1/3$	$-1/3$	$\bar{d}_R$	$+2/3$	0	$+1/3$	$+1/3$
e_L	-1	$-1/2$	$-i/\sqrt{12}$	$+i/\sqrt{12}$	$\bar{e}_L$	$+1$	$+1/2$	$+i/\sqrt{12}$	$-i/\sqrt{12}$
e_R	-2	0	0	0	$\bar{e}_R$	$+2$	0	0	0
u_L	$+1/3$	$+1/2$	$+1/6$	$+1/6$	$\bar{u}_L$	$-1/3$	$-1/2$	$-1/6$	$-1/6$
u_R	$+4/3$	0	$-1/3$	$-1/3$	$\bar{u}_R$	$-4/3$	0	$+1/3$	$+1/3$

(18)

Thus we obtain exactly the quantum numbers of the standard model elementary fermions.

6 Discussion

The previous paper [4] examined single particle spin path integrals over mutually unbiased bases (MUBs) of the Pauli (spin-1/2) algebra. That paper showed that the generation structure of the elementary fermions is naturally associated with the long term propagators of those single particle spin path integrals. The present paper extends the analysis to multiparticle spin path integrals and obtains the weak hypercharge and weak isospin quantum numbers in a natural manner.

The justification for examining spin path integrals over MUBs is that position path integrals can be thought of as products of MUBs [1]. Extending this definition to spin path integrals leads naturally to the calculations of this paper and the previous.

The generation calculations of the previous paper and the weak quantum number calculations of the present paper are independent in that the weak quantum number calculations do not depend on generation. This is necessary to obtain three identical generations.

In both the case of the generations and the weak quantum numbers, the solutions were found as a result of restrictions which amount to projection operators (or idempotency). The generation quantum number entered into the previous paper in the form of $\exp(2ig\pi/3)$ factors. A generalization of g from an integer to a complex number can be interpreted as a change in the relative phases of the propagators. Such a change causes the propagator to lose its idempotency but it makes physical sense. Thus we can imagine that superpositions of particles from different generations can occur in nature as is seen in the weak mixing matrices.

On the other hand, the weak quantum number calculations of the present paper arise from topological differences in the long term multi-particle propagators. In algebraic quantum field theory, topological quantum numbers lead to superselection sectors [5]. So the present paper's topological calculation for the weak quantum numbers gives a good explanation for why nature does not allow the superposition of an electron with a quark, but does allow the superposition of particles from different generations.

Quantum information theory examines the information content of quantum mechanics. A basic concept of quantum mechanics is the Heisenberg uncertainty principle. MUBs extract the information content of this central principle. In this paper we've examined the simplest possible MUB system, that of spin-1/2, and found that the structure of the elementary fermions appears to be related to this system. This is reminiscent of how spin-1/2 is the simplest possible angular momentum system. In a certain sense, these two papers simply combine Pauli's analysis of angular momentum with Heisenberg's analysis of the relationship between position and momentum. These are very basic ideas in quantum mechanics, it cannot be surprising that they are worth understanding.

7 Acknowledgments

The author thanks his parents for financial assistance and Marni Sheppeard for advice and encouragement.

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