The discrete Fourier transform and the particle mixing matrices

C. A. Brannen*
Liquafaction Corporation
720 Road N NE,
Moses Lake, WA 98837 USA

M. D. Sheppeard
Oxford University Computing Laboratory
Wolfson Building, Parks Rd
Oxford, OX1 3QD, UK
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In quantum mechanics, the Fourier Transform commonly converts from position space to momentum. For finite dimensional Hilbert spaces, the analog is the discrete (or *quantum*) Fourier transform, which has many applications in quantum information theory. We explore applications of this discrete Fourier transform to the elementary particle generations, and then present a related and elegant new parameterization for unitary 3×3 matrices that is compatible with the tribimaximal MNS matrix.

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I. INTRODUCTION

In quantum mechanics, the Fourier transform takes position space to momentum space. These observables are complementary, or *unbiased*. That is, a state which is an eigenstate of position has no information about its momentum and vice versa. For finite dimensional Hilbert spaces, the discrete (or quantum) Fourier transform [1] converts a basis into a complementary basis. Here we will mostly be concerned with the discrete Fourier transform $F(\mathbf{v}) = \tilde{\mathbf{v}}$ of vectors $\mathbf{v}$ from a three dimensional Hilbert space, that is, the qutrit space. The transform may be written using a unitary matrix F:

$$\tilde{\mathbf{v}} = F\mathbf{v} = \frac{1}{\sqrt{3}} \begin{pmatrix} \omega & \omega^* & 1 \\ \omega^* & \omega & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}. \qquad (1)$$

where $\omega = \exp(2i\pi/3)$ is the complex cubed root of unity. Given a set of basis vectors for the Hilbert space $\{\mathbf{v}^k\}$, the Fourier transform converts this basis set into a new basis set $\{\tilde{\mathbf{v}}^k\}$ that is unbiased with respect to the original basis set.

One can also consider collections of basis sets for a finite dimensional Hilbert space, each of which is unbiased with respect to the others. Such a collection is called "mutually unbiased". The maximum possible size of such a collection, for general dimension n, is an important unsolved problem in quantum information theory. For n prime, or the power of a prime, it is known that collections containing $n+1$ mutually unbiased basis sets exist.

The Fourier transform of state vectors is associated with a transformation of linear operators O, given by

$$\tilde{O} = FOF^{-1}. \qquad (2)$$

Since the Fourier transform matrix F is unitary, the Fourier transform of a unitary matrix is unitary. The columns of a unitary matrix can be considered as a set of basis vectors for the Hilbert space and vice versa. Consequently, a complete set of mutually unbiased bases for the Hilbert space of dimension n defines a set of $n+1$ unitary operators.

For $n = 2$, a complete set of mutually unbiased bases is given by $\{F_2, Y, Y^2 = I\}$ where F_2 is the 2×2 Fourier transform matrix:

$$F_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad (3)$$

which happens to be equal to $(\sigma_x + \sigma_z)/\sqrt{2}$, and $Y = (1+\sigma_y)/\sqrt{2}$ which is $\sqrt{2}$ times the density matrix for spin in the y direction. Note that Y is a unitary square root of the identity I. There are other Hadamard type operators [2] that also transform a basis set into a mutually unbiased basis set. The collection of all these operators forms a finite set, in principle characterising the measurable quantities for quantum mechanics in a given dimension. For dimension 3, a complete set of four mutually unbiased bases is given by the collection $\{F, R, R^2, R^3 = I\}$ where R is

$$R = \frac{1}{3} \begin{pmatrix} 1 & \omega & 1 \\ 1 & 1 & \omega \\ \omega & 1 & 1 \end{pmatrix}, \qquad (4)$$

a unitary cubed root of the identity.

*Electronic address: Carl@Brannenworks.com

II. EXPERIMENTAL EQUIVALENCE CLASSES

Experimentally, the CKM and MNS matrix elements can only be measured in the form of squared magnitudes (probabilities). A matrix of probabilities contains much less information than a unitary matrix. Consequently, while a parameterization of the unitary 3×3 matrices require nine parameters, one can parameterize the squared magnitudes of such a matrix with only four parameters. We will call two unitary matrices *equivalent* if one can be obtained from the other by multiplication of rows and columns by complex phases. This relation defines equivalence classes of matrices. For a given probability matrix, the number of different equivalence classes that give that probability matrix depends on the symmetry of the probability matrix.

The CKM matrix entries are approximately in $2+1$ block diagonal form. Squaring recent estimates [3] of the CKM amplitudes we have:

$$P_{CKM} = \left(\begin{array}{cc|c} 0.9483(5) & 0.0516(5) & 0.000016(1) \\ 0.0516(5) & 0.9467(5) & 0.00178(7) \\ \hline 0.00007(1) & 0.00173(6) & 0.99820(7) \end{array}\right) \quad (5)$$

The $2+1$ block diagonal unitary matrices are easy to parameterize. First let θ be a real angle, and let α, β, γ and η be four complex phases. Define $U(\theta,\alpha,\beta,\gamma)$ as

$$U(\theta,\alpha,\beta,\gamma) = \begin{pmatrix} \alpha\gamma\cos(\theta) & -\alpha\sin(\theta) & 0 \\ \beta\gamma\sin(\theta) & \beta\cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6)$$

Then a complete parameterization of the $2+1$ block diagonal matrices is given by $\eta\ U$ where η is an overall complex phase.

While the usual parameterizations of the CKM matrix use four mixing angles and one complex phase, the above parameterization of the $2+1$ block matrices uses four complex phases and one mixing angle. Since the phase η is just an overall complex phase, we will set $\eta = 1$ and consider a parameterization with four variables: three complex phases α, β and γ, and one mixing angle θ.

Since F is a unitary matrix, $\tilde{U}(\theta,\alpha,\beta,\gamma)$ is also a unitary matrix. It is also *magic*, which is to say that the elements of a row or column sum to the same number, one. This is reminiscent of the unitary property of the CKM matrix, that the sum of the squared magnitudes of the elements of each row or column sum to one.

In general, 3×3 complex (unitary) matrices require nine complex (real) parameters. Similarly, the $2+1$ block diagonal complex (unitary) matrices require five complex (real) parameters.

Given $\tilde{U}(\theta,\alpha,\beta,\gamma)$ we can define a parameterization of unitary matrices by multiplying the rows and columns of $\tilde{U}$ by complex phases. There are three rows and three columns, but multiplying all rows by a complex phase is equivalent to multiplying all columns by the same phase so there is a reduction by one degree of freedom. This gives a full parameterization of unitary 3×3 matrices of the form

$$D_r F\ U(\theta,\alpha,\beta,\gamma)\ F^{-1} D_c \quad (7)$$

where D_r and D_c are diagonal matrices that give the complex phases multiplying rows and columns.

III. CIRCULANT MATRICES

An m-circulant matrix is defined by its first row; the other rows being equal to the first except for a shift of each entry m places to the right. The discrete Fourier transform diagonalises 1-circulants, and a two dimensional Fourier transform similarly diagonalises a 1-circulant 2×2 matrix, which corresponds to permutations of two objects. In dimension three, 2-circulant matrices are transforms of the off diagonal $2+1$ block matrices.

A 3×3 m-circulant may be characterized by the three complex numbers that are the entries of its first row. Any magic 3×3 matrix can be written as the sum of a 1-circulant and a 2-circulant matrix. This defines a natural decomposition for the magic unitary matrices, but such a decomposition is not quite unique.

By choosing the overall complex phase appropriately, we can arrange for an arbitrary 3×3 magic unitary matrix to be written as the sum of a real 1-circulant matrix and an imaginary 2-circulant matrix. Let a, b, c, and d be four real numbers. Define the six real numbers I, J, K, R, G, and B as follows:

$$\begin{array}{rcl} I &=& [\cos(a) + 2\cos(b)\cos(c+0\pi/3)]/3, \\ J &=& [\cos(a) + 2\cos(b)\cos(c+2\pi/3)]/3, \\ K &=& [\cos(a) + 2\cos(b)\cos(c+4\pi/3)]/3, \\ R &=& [\cos(a) + 2\cos(b)\cos(d+0\pi/3)]/3, \\ G &=& [\cos(a) + 2\cos(b)\cos(d+2\pi/3)]/3, \\ B &=& [\cos(a) + 2\cos(b)\cos(d+4\pi/3)]/3. \end{array} \quad (8)$$

Then there is a magic unitary matrix W

$$W = \begin{pmatrix} I & J & K \\ K & I & J \\ J & K & I \end{pmatrix} \pm i \begin{pmatrix} R & B & G \\ B & G & R \\ G & R & B \end{pmatrix}, \quad (9)$$

which defines a parameterization of the 3×3 magic unitary matrices (and therefore the equivalence classes of unitary matrices). A parameterization of all unitary 3×3 matrices is provided by $D_r\ W\ D_c$. Note that the nine entries of W are all possible sums of one element from $\{I,J,K\}$ with one element from $\{iR,iG,iB\}$.

Since the complex numbers on the top row of an m-circulant define the whole matrix, one can think of the m-circulants as forming a complex n-vector space. The basis elements for the m-circulants are permutation matrices, that is, they are matrices of zeroes and ones that define a permutation on the components of n-vectors. With respect to Eq. (8), the elements I, J, and K correspond to the even permutations on three elements, while R, G, and B correspond to the odd permutations.

Since the 1-circulant component is the real part of the unitary matrix, while the 2-circulant is the imaginary part, their contributions to the matrix of probabilities do not have any interference. That is, there is a natural decomposition of the probability matrix into the sum of a real 1-circulant matrix and a real 2-circulant matrix. This decomposition uses three probabilities from the 1-circulant matrix and three more from the 2-circulant. The probabilities for the full unitary matrix correspond to the nine possible results obtained by adding one of the three 1-circulant probabilities, $\{I^2, J^2, K^2\}$, to one of the three 2-circulant probabilities, $\{R^2, G^2, B^2\}$.

IV. THE MNS LEPTON MIXING MATRIX

Present measurements of the MNS matrix probabilities are in agreement with the tribimaximal values:

$$P_{MNS} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}. \qquad (10)$$

By choosing the overall complex phase appropriately, we can write a representative from an associated equivalence class of unitary matrices as the sum of two circulants, one real and one imaginary. For the MNS matrix, one may choose:

$$U_{MNS} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & 1 & 0 \\ 0 & \sqrt{2} & 1 \\ 1 & 0 & \sqrt{2} \end{pmatrix} \pm \frac{i}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & -1 & 0 \\ -1 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & -1 \end{pmatrix}. \qquad (11)$$

Thus the tribimaximal probabilities may be characterized using the nine possible results obtained by selecting one element of $\{1/3, 1/6, 0\}$ and adding it to an element of $\{1/3, 1/6, 0\}$.

The Fourier transform of U_{MNS} also takes a particularly simple form:

$$\begin{aligned} \tilde{U}_{MNS} = \tfrac{1}{\sqrt{6}} & \begin{pmatrix} \sqrt{2} & -1 & 0 \\ -1 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2}-1 \end{pmatrix} \\ + \tfrac{1}{\sqrt{6}} \begin{pmatrix} \omega^* & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix} & + \tfrac{1}{\sqrt{6}} \begin{pmatrix} 0 & \sqrt{2}\omega^* & 0 \\ \sqrt{2}\omega & 0 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}, \end{aligned} \qquad (12)$$

involving cubed root phases ω, rather than the phases i that appear in the more symmetric representation used in Eq. (11). In other words, the Fourier transform reflects the difference between the root systems [4] for (a) $su(2) \times u(1) \subset su(2) \times su(2)$ and (b) $su(3)$. The diagonal matrix, with the cubed roots of unity, is the standard Weyl generator for the qutrit torus, namely the three dimensional generalization of the Pauli matrix σ_z. The discrete Fourier transform $F(\tilde{U}_{MNS})F^{-1}$ takes this diagonal to a permutation (231), that is, the K part of the matrix given in Eq. (8).

Define the Fourier transform for the $2+1$ block diagonal matrices as:

$$F_{21} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}. \qquad (13)$$

This is a unitary matrix, and $F\ F_{21}$ is also a unitary matrix with tribimaximal entries, although permuted from the usual form. This factorisation is in line with the A_4 discrete symmetry [5] studied in the context of neutrino mass generation in the standard model, since this group is the product $\mathbb{Z}_3 \ltimes \mathbb{Z}_2$ of cyclic permutation groups. Here, however, the finite group appears as a discrete structure in quantum information theory, rather than as a subgroup of a large continuous symmetry.

An alternative representation of the tribimaximal mixing matrix is the product RF_{21}, where R is the other 3×3 complementary observable automorphism [2]. Essentially, the only other product involving the mutually unbiased operators is of the form:

$$\begin{aligned} FR_2 \equiv \frac{1}{\sqrt{6}} & \begin{pmatrix} \omega & \omega^* & 1 \\ \omega^* & \omega & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & i & 0 \\ i & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \qquad (14) \\ = \frac{1}{\sqrt{6}} & \begin{pmatrix} \omega + i\omega^* & i\omega + \omega^* & \sqrt{2} \\ i\omega + \omega^* & \omega + i\omega^* & \sqrt{2} \\ 1+i & 1+i & \sqrt{2} \end{pmatrix}, \end{aligned}$$

which yields the probability matrix

$$\|FR_2\|^2 = \frac{1}{3} \begin{pmatrix} 1-x & 1+x & 1 \\ 1+x & 1-x & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad (15)$$

where $x = \sqrt{3}/2$. The product of (a permuted form of) P_{MNS} and $\|FR_2\|^2$ returns another permuted form of P_{MNS} when $x = 1/2$. In general, for different values of x, this product

$$\frac{1}{3} \begin{pmatrix} 1+x & 1-x & 1 \\ 1+x & 1-x & 1 \\ 1-2x & 1+2x & 1 \end{pmatrix} \qquad (16)$$

describes all magic probability matrices of P_{MNS} type, namely with one unbiased column equal to $(1/3, 1/3, 1/3)$ and two rows with an unbiased pair. A permitted small deviation from $x = 1/2$ corresponds to the $\sin\theta_{13}$ term in the literature [6].

V. MASS MATRICES

In [7] the Koide [8] formula:

$$2(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2 = 3(m_e + m_\mu + m_\tau) \qquad (17)$$

for charged lepton masses was recovered in the form of a 3×3 circulant complex matrix. This analysis has been extended successfully to a set of three neutrinos. These mass matrices take the form:

$$M = \eta \begin{pmatrix} 1 & re^{i\theta} & re^{-i\theta} \\ re^{-i\theta} & 1 & re^{i\theta} \\ re^{i\theta} & re^{-i\theta} & 1 \end{pmatrix} \tag{18}$$

for real η, r and θ. The eigenvalues are given by

$$\lambda_k = \eta[1 + 2r\cos(\theta + 2\pi k/3)], \tag{19}$$

where $m_i = \lambda_i^2$ are the rest mass values. The Koide formula results from setting $r^2 = \frac{1}{2}$, and this choice may be applied also to the neutrino matrix.

In general, the 3×3 Fourier transform takes a diagonal mass matrix with entries (m_1, m_2, m_3) to the 1-circulant matrix with top row elements

$$\begin{gathered} (m_1 + m_2 + m_3)/3, \\ (m_1\omega + m_2\omega^2 + m_3)/3, \\ (m_1\omega^2 + m_2\omega + m_3)/3. \end{gathered} \tag{20}$$

VI. CONCLUSIONS

In summary, the special operators associated to sets of mutually unbiased bases [2] in quantum mechanics may be used to define an elegant parameterization of 3×3 unitary matrices and their probability matrices. These operators are also associated with circulant mass matrices.

Although the elements of the MNS matrix are particularly simple from this point of view, the CKM matrix is more complicated. Given that the low energy domain better approximates a weak coupling limit, one does not expect quark mixing to be as straightforward as neutrino mixing.

Observe also that, in quantum information theory, the operators appearing here act *locally* on states for Hilbert spaces of dimension two (qubit) or three (qutrit), where several qubits or qutrits may be combined using the tensor product of Hilbert spaces. The nonlocal information content of the CKM matrix may involve mixing in higher dimensional state spaces, such as the 27 dimensional space for three qutrits. Such ternary states are reducible to 3×3 form when state amplitudes are expressed in terms of certain projective coordinates [9].

This work is partly motivated by a modern diagram calculus for finite dimensional quantum mechanics [11], in which an observable is described by specific rules on graph vertices and edges, and where the introduction of colored vertices permits simple rules for complementary observables.

Automorphisms such as the Fourier transform interact nicely with the vertices that represent basis structures. Quantum information circuits using the Fourier transform may be expressed simply in diagrammatic form. Underlying these diagrams is the abstract concept of a monoidal category, now common in many areas of physics, such as the analysis of two dimensional condensed matter systems exhibiting anyonic behaviour.

In other words, the discrete Fourier transform may be linked to highly abstract concepts being used to discuss mass generation in approaches to particle physics that do not rely solely on classical symmetry principles.

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