Spinors, Gauges, Geometric Algebra, and Probabilities

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Recent advances[1] in Bohmian mechanics suggest that quantum mechanics should consider the density matrix as the fundamental quantum object rather than the spinor wave state. This change allows a theory that is invariant with respect to the $\mathcal{U}(1)$ gauge, but still reproduces the results of gauge theory such as the Aharonov-Bohm effect; and in a simpler manner. This advance amounts to retreating from a formalism based on Hilbert space to one based on a Banach space. For a given Banach space there are many Hilbert spaces that it can be factored into. The choice of Hilbert space produces an unphysical gauge freedom.

This paper applies Hestenes' Geometric Algebra (GA) to density matrices and generalizes to multiple particles by using Schwinger's elegant "measurement algebra" (SMA). Doing this eliminates the arbitrary orientation in Hestenes' version of the Dirac equation. More importantly, this allows us to produce a theory that is gauge invariant with respect to the more general gauges associated with forces other than $\mathcal{U}(1)$.

In moving from a Hilbert space to a Banach space, one loses the automatic use of complex numbers in defining probabilities. We show that there is an alternative probability interpretation that is purely geometric and for the usual representations of Clifford algebras gives identical values to the usual spinor probabilities of quantum mechanics. We also show interesting representations where the two methods of defining probabilities differ.

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A 2004 paper by Brown and Hiley[1] describes the Bohmian mechanics of density matrices. The older theory of Bohmian mechanics was tied to the x-representation. By using density matrices and the idempotency relation:

$$\mu\,\mu = \mu, \tag{1}$$

the new theory shows that the Bohmian interpretation is general and is not tied to any particular representation. Thus Bohmian mechanics now follows the full symplectic symmetry of quantum mechanics. For a discussion of the various ways density matrices appear in Bohmian mechanics, see [2], which suggests that spin requires that Bohmian mechanics use density matrices.

An important advantage of the density matrix version of Bohmian mechanmics is that it uses elements that are ($\mathcal{U}(1)$) gauge invariant and reproduces the results of gauge theory, such as the Aharonov-Bohm phase, in a very simple way.[1] While Brown and Hiley concentrate on position and momentum considerations, in this paper we will instead be concentrating on spin and internal symmetries.

In the 1980s, David Hestenes wrote the Dirac wave equation in fully geometric form using his "Geometric Algebra" (GA). The imaginary number i of the usual spinor form of the Dirac equation is replaced by an arbitrary geometric orientation (bivector) in Hestenes' theory. While this orientation (or type of gauge) does not appear in the predictions of the theory, some physicists have found it inelegant.[3] We show (trivially) that on passing to the density matrix form, Hestenes' Dirac equation loses its arbitrary orientation. Following the method of Trayling [4] we use ideals of Clifford algebras to represent different particles.

In the 1950s and 60s, Julian Schwinger developed[5] an elegant general scheme for quantum kinematics and dynamics appropriate to systems with a finite number of dynamical variables, now known as "Schwinger's Measurement Algebra" (SMA). The "selective measurements", $M(a')$ of the SMA are particularly elegant in that they possess no unphysical gauge freedom. A particular measurement of this sort can possess a unique representation in the GA. On the other hand, we will show that Schwinger's "general selective measurements" $M(a', b')$, when written in the GA, will have an arbitrary orientation of the same sort seen in Hestenes' Dirac equation.

In avoiding the arbitrary orientation of Schwinger's general selective measurements, we will find reason to suspect that the splitting of measurements into bras and kets may be unphysical and a mathematical convenience only. This leads us to doubt the use of complex numbers in Born's probability postulate. In density matrix theory, the use of complex numbers is replaced by the use of the trace, but this is also, in general, a complex number.

As a replacement for the use of complex numbers in quantum mechanics, we find that the natural Banach space defined by a Clifford algebra as a vector space over its canonical basis elements allows us to calculate probabilities without recourse to the traditional factoring of the Banach space of density matrices into a Hilbert space of spinors or the use of a complex valued trace. For a calculation showing this equivalence for the example of the Pauli algebra see the Appendix A. We show that the

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equivalence holds for the usual representations of Clifford algebras used in physics such as the Pauli and Dirac algebras, but that any nontrivial Clifford algebra can be put into a matrix representation where the probabilities are not equivalent. The calculation relies on a characterization of the matrix representations of Clifford algebras that is based on primitive idempotents and is very efficient and easy to use.

The usual spinor probability interpretation does well at modeling the individual particles of the Standard Model. We show that the application of the modified probabilities contemplated here is to the relationships between different particles. Accordingly, we provide some relationships between complex phase and probability that may be useful in characterizing representations of Clifford algebras beyond those used in the Standard Model.

I. ORGANIZATION OF THE PAPER

Section (II) is a discussion of the orientation gauge of Hestenes' version of the Dirac equation. It is easily shown that this orientation gauge disappears when the wave function is converted to density matrix form. This section is written in the usual form of the Dirac algebra, gamma matrices. In order to work with more general Clifford algebras, we include Section (III) to introduce the notation and provide some exercises for the reader unfamiliar with Clifford algebra.

For density matrices, we will need only the simplest parts of Schwinger's measurement algebra[5], these are introduced in Section (IV), where we also show that the SMA is compatible with the density matrix formalism (and therefore provides a multiparticle generalization of the density matrix). In Section (V) we introduce Schwinger's "general measurement" symbols and show that despite their apparent simplicity and generality, they nevertheless imply a choice of orientation gauge.

"Square spinors" are how one defines a spinor in the algebra of operators. In the usual Pauli or Dirac representations, this amounts to packing the spinor into one column of a matrix, thus putting both the operators and spinors into the same, matrix, form. Square spinors are the subject of Section (VI). With square spinors introduced, we can derive the relationship between the trace function defined in the SMA and the natural squared magnitude defined on a Clifford algebra, and this is the subject of Section (VII).

In Section (VIII) we consider products of projection operators in the Pauli algebra. Physically, these correspond to sequences Stern-Gerlach filters. We derive a simple method for calculating these products and show some symmetries. Section (IX) analyzes interference between sequences of Pauli type Stern-Gerlach filters.

Section (X) derives a relationship between the probabilities and phases of products of three Pauli type filters, and generalizes this result for other Clifford algebras and the new probability interpretation. Section (XI) demonstrates how to obtain non standard representations of Clifford algebras by using exponentials, and applies these to toy models of elementary particles and their interactions.

Section (XII) shows how the formula $P_\theta = (1 + \cos(\theta))/2$ changes when the Pauli algebra is altered so that the spinors share a part from a hidden dimension of either $+1$ or -1 signature. Section (XIII) introduces a replacement for complex phases in the context of the unusual representations of Clifford algebras used here. Section (XIV) collects together the information about the projection operators of the Pauli algebra and generalizations of the Pauli algebra from the previous 6 sections into a practical set of rules.

Section (XV) that in the context of the Schwinger Measurement algebra modeled in a Clifford algebra, the elementary fermions may be modeled as composite particles made up of 3 preons (subparticles) each. Section (XVI) derives the relationship between the SMA of a preon and the SMA of the deeply bound composite particles composed of preons. We define a natural form for cross generation operators.

Section (XVII) it is shown that the experimentally measured masses of the charged leptons can be produced by a particularly simple operator. We generalize this to the masses of the quarks. In Section (XIX), we use the quark and charged lepton masses to predict the masses of the neutrinos.

Section (XX) wraps up the conclusions of the paper and Section (XXI) acknowledges assistance given to the author.

II. ORIENTATION IN HESTENES' DIRAC EQUATION

When the Dirac equation is written in Hestenes' Geometric Algebra:[1]

$$-\nabla\Psi_H\ \gamma_1\gamma_2 = m\Psi_H\ \gamma_0, \quad \Psi_H \in \mathcal{CL}^+_{1,3} \tag{2}$$

there is an orientation assigned to the wave equation and to the wave values. In the above, the orientation chosen is $\gamma_1\gamma_2$. This "bivector" defines the geometric equivalent of the imaginary number i of the usual Dirac equation. Of course this orientation does not appear in the physical predictions of the theory; it is only a gauge freedom. For a discussion of this gauge issue in the context of the choice of signature, see Pezzaglia's 1997 paper [9, §V]. A very useful short paper by William E. Baylis is worth quoting extensively:

4* Symmetry of the Hestenes Equation

[1] For example, compare [6, 10.2] with [7, Eq (166)] or [8, Eq (53)].

Part of Joyces stated reason for seeking an alternative algebraic form of the Dirac equation was that he viewed the Hestenes form (*) as giving special status to given directions in space. In particular, because equation (*) contains the $\gamma_1\gamma_2$ bivector, it was felt that the corresponding plane was singled out. On this basis it might be disappointing that every solution to the Joyce equation (*) is a combination of solutions to two equations of the Hestenes form. However, the asymmetry that Joyce saw in the Hestenes equation is only apparent, as explained below.

The principal advantages of the Hestenes formulation of the Dirac equation are (1) that it acts in the real Dirac algebra $\mathcal{CL}_{1,3}$ rather than in the more traditional complex Dirac algebra used by most authors as well as by Joyce, and (2) it offers unambiguous geometrical interpretations for expressions in the theory. The fact that the spinor of the Hestenes formulation is an even element and that the Hestenes equation preserves its evenness suggests that the Dirac theory can also be formulated in the real Pauli algebra $\mathcal{CL}_3$, which is isomorphic to $\mathcal{CL}^+_{1,3}$. Indeed there is a very simple covariant formulation[*][*][*] using paravectors[*] of $\mathcal{CL}_3$, and this is closely related to formulations in biquaternions and 2×2 matrices.[*][*][*] Further background and references can be found in a couple of recent papers.[*][*]

In both the $\mathcal{CL}_3$ formulation and in Hestenes analysis, the spinor plays the role of a relativistic transformation amplitude from the reference frame of the fermion to the lab frame. The orientation of the reference frame is not significant since global gauge transformations $\Psi_H \rightarrow \Psi_H R$, where R is a fixed spatial rotor, can rotate it arbitrarily. It is therefore of no physical consequence that the particular bivector $\gamma_1\gamma_2$ appears in the Hestenes equation (*).[*] [3, §4]

The density matrix form is itself inherently free of the orientation issue of the spinor form. To change from one orientation for ψ to another one uses the transformation $\psi_H \rightarrow \psi_H R$ as mentioned above by Baylis. But $RR^\dagger = 1$ and therefore the density matrix does not depend on orientation:

$$\rho = \psi_H \psi_H^\dagger \rightarrow \psi_H R R^\dagger \psi_H^\dagger = \rho. \tag{3}$$

Schwinger's measurement algebra appears to be immune to the orientation issue but in the next section we will show that this is the case only if one avoids his "general measurements".

It is interesting that if one considers the elementary particles to be the handed, or chiral states, so that the appropriate Dirac equation is massless, the orientation of Hestenes' Dirac equation can also be removed. Following the notation of the Baylis paper cited above, Hestenes' Dirac equation, Eq. (2) becomes

$$-\nabla\Psi_H\ \gamma_1\gamma_2 = 0, \quad \Psi_H \in \mathcal{CL}^+_{1,3} \tag{4}$$

and one can right multiply by $\gamma_2\gamma_1$ to give

$$-\nabla\Psi_H\ = 0, \quad \Psi_H \in \mathcal{CL}^+_{1,3}. \tag{5}$$

Thus there is no orientation issue here at least at the preon level. That the orientation issue arises only with the introduction of mass suggests that mass, as an interaction between preons, has to do with orientation.

Another argument for preferring a density matrix formalism to spinors is that ensembles can be described in density matrix form in a particularly natural way, and there is a simple relationship between the entropy of a quantum ensemble in the density matrix form:

$$S = -k\,\mathrm{tr}(\iota_A \ln(\iota_A)), \tag{6}$$

where k is Boltzmann's constant. Also, the equation of motion for the density matrix can be written in the form for Liouville's equation in classical statistical mechanics if one modifies the Hamiltonian appropriately:

$$\partial_t\ \iota = -[\iota, H/i\hbar]. \tag{7}$$

III. CLIFFORD ALGEBRAS AND NOTATION

In this article we will frequently have cause to work out examples using the Pauli or Dirac algebras. Instead of using a specific matrix form for these algebras we will instead use "hat" notation to indicate the geometric content of the elements. For this we will here introduce the notation for the Pauli and Dirac algebras. Rather than treating these as specific matrices, we will instead treat them as elements of Clifford algebras as we now explain.

The Pauli algebra will be defined by the three $\mathcal{CL}(3,0)$ Clifford algebra vectors $\hat{x}$, $\hat{y}$ and $\hat{z}$ (that correspond to the usual matrices σ_x, σ_y and σ_z), along with all possible products, and all sums multiplied by real numbers. These three vectors each square to 1 and anticommute among themselves:

$$\begin{gathered}\hat{x}\hat{x} = \hat{y}\hat{y} = \hat{z}\hat{z} = 1, \\ \hat{x}\hat{y} = -\hat{y}\hat{x}, ...\end{gathered} \tag{8}$$

In using x, y, z notation, we do not intend on requiring that the canonical basis vectors be associated with the usual spatial dimensions. The Pauli algebra also arises in internal symmetries such as isospin. The set of all possible products includes eight elements, and these form the basis for the Pauli algebra over the reals. We use "wide hats" to indicate that these products individually have geometric interpretations. For example, $\hat{x}\hat{y} = \widehat{xy}$

indicates a rotation in the $x-y$ plane. The eight possible products are:

$$\begin{array}{cccc} \hat{1}, & \hat{x}, & \hat{y}, & \hat{z}, \\ \widehat{xyz}, & \widehat{yz}, & \widehat{xz}. & \widehat{xy} \end{array} \tag{9}$$

When $\hat{1}$, $\hat{x}$, $\hat{y}$ and $\hat{z}$ are replaced by the 2×2 unit matrix and the Pauli spin matrices, respectively, the above eight products provide a basis set for the 2×2 complex matrices over the reals. That is, any 2×2 complex matrix $\hat{\alpha}$ can be written as a sum:

$$\hat{\alpha} = \alpha_1\hat{1} + \alpha_x\hat{x} + ... + \alpha_{xyz}\widehat{xyz}, \tag{10}$$

with α_χ taken from the reals. The element $\widehat{xyz}$ is special. In the Clifford algebra, it is the unit pseudoscalar. In the Pauli matrices, it is the imaginary unit matrix. In either case it commutes with all the other elements of the algebra and squares to -1. We will not write $\widehat{xyz}$ as i, but instead will reserve i for its traditional use as a square root of -1.

For the Dirac algebra, we will use the $(-,+,+,+)$ signature. The correspondence of the $\mathcal{CL}(3,1)$ Clifford algebra vectors and the traditional gamma matrices is:

$$\hat{x} \equiv \gamma_1, \quad \hat{y} \equiv \gamma_2, \quad \hat{z} \equiv \gamma_3, \quad \text{and} \quad \hat{t} \equiv \gamma_0. \tag{11}$$

The vectors square to the signature and anticommute:

$$\begin{array}{c} \hat{x}\hat{x} = \hat{y}\hat{y} = \hat{z}\hat{z} = -\hat{t}\hat{t} = 1 \\ \hat{x}\hat{y} = -\hat{y}\hat{x}, ... \end{array} \tag{12}$$

Using the above set of anticommutation and squaring equations, any product of the vectors can be reduced to $\pm$ one of sixteen possible products. As before, we denote these with wide hats. The first eight are identical to the Pauli algebra products, the rest include factors of $\hat{t}$:

$$\begin{array}{cccc} \hat{1}, & \hat{x}, & \hat{y}, & \hat{z}, \\ \widehat{xyz}, & \widehat{yz}, & \widehat{xz}. & \widehat{xy} \\ \hat{t}, & \widehat{xt}, & \widehat{yt}, & \widehat{zt}, \\ \widehat{xyzt}, & \widehat{yzt}, & \widehat{xzt}. & \widehat{xyt} \end{array} \tag{13}$$

When the canonical basis vectors $\hat{x}$, $\hat{y}$, $\hat{z}$ and $\hat{t}$ are replaced by some specific choice of the Dirac gamma matrices, the above sixteen products provide a *complex* basis set for the complex 4×4 matrices. That is, any 4×4 complex matrix $\hat{\alpha}$ can be written as a sum:

$$\hat{\alpha} = \alpha_1\hat{1} + \alpha_x\hat{x} + ... + \alpha_{xyzt}\widehat{xyzt}, \tag{14}$$

where α_χ are *complex* numbers. The element $\widehat{xyzt}$ is special. In the Clifford algebra it is the unit pseudoscalar and squares to -1. Many other elements of the algebra also square to -1 such as, for example, $\widehat{yz}$ or $2.6\hat{x} - 2.4\hat{t}$. But no real element of the algebra squares to -1 *and* commutes with all the other elements. In addition, no element commutes with all the other elements except $\hat{1}$ and its multiples.

The product of the three Pauli matrices:

$$\begin{array}{rl} \hat{x}\hat{y}\hat{z} = & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ = & i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \tag{15}$$

is the imaginary unit so it is natural in the Pauli algebra to use complex numbers. But there is no product of gamma matrices that is the imaginary unit. Instead, to write the imaginary unit over the gamma matrices, one must take the imaginary multiple of the unit matrix. This is equivalent to the statement that the real Clifford algebra $\mathcal{CL}(3,1)$ does not contain any elements that square to -1 and commute with the algebra.

We will also use examples taken from the Clifford algebra $\mathcal{CL}(4,1)$. We will designate the canonical basis vectors as $\hat{x}$, $\hat{y}$, $\hat{z}$, $\hat{s}$ and $\hat{t}$. As usual, these square to the appropriate signature and anticommute:

$$\begin{array}{c} \hat{x}^2 = \hat{y}^2 = \hat{z}^2 = \hat{s}^2 = -\hat{t}^2 = 1 \\ \hat{x}\hat{y} = -\hat{y}\hat{x}, ... \end{array} \tag{16}$$

The possible products of the 5 basis vectors gives $2^5 = 32$ distinct products, $\hat{1}, \hat{x}, ...\widehat{xyzst}$, and these 32 products form a real basis for the Clifford algebra $\mathcal{CL}(4,1)$. The unit pseudoscalar, $\widehat{xyzst}$, commutes with all the elements of the algebra and squares to -1, and so forms a geometric imaginary unit i. However, note that we will not write i for $\widehat{xyzst}$, and will instead reserve i for its traditional mathematical use as a non real root of the equation $x^2 = -1$.

When we consider complex Clifford algebras, we may treat i as if it were a canonical basis vector of the algebra. For example, 2.5 $\widehat{ixy}$ is to signify $2.5i$ $\widehat{xy}$. As an algebraic element, the effect is to add i to the list of canonical basis vectors subject to the additional algebraic rules that it squares to -1 and commutes with the other vectors:

$$\hat{i}\hat{i} = -1, \qquad \hat{i}\,\hat{x} = \hat{x}\,\hat{i}..., \tag{17}$$

where χ is any other canonical basis vector (or, indeed, any element of the algebra). The inclusion of $\hat{i}$ among the canonical basis vectors doubles the number of products.

Some example calculations may be useful:

$$\begin{array}{l} \widehat{xyz}\ \widehat{yzt} = -\widehat{xt}, \\ \widehat{xyzst}\ \hat{y} = -\widehat{yst}, \\ \widehat{xt}\ \widehat{zt} = \widehat{xz}, \\ \widehat{xyzs}\ \widehat{xyzs} = 1, \\ \widehat{ixy}\ \widehat{ixy} = 1, \\ (\cos(\theta)\hat{x} + \sin(\theta)\hat{y}\ )^2 = 1, \\ (\cosh(\theta)\widehat{it} + \sinh(\theta)\widehat{xyz})^2 = 1, \end{array}$$

where θ is any real number. The first three of the above are examples of simple products of canonical basis vectors. The next four are examples are "roots of unity". We will follow the mathematical literature and refer to

elements that satisfy $\iota^2 = \iota$ as "idempotent", unless they have an immediate physical interpretation as projection operators or measurements. Four examples of idempotents:

$$\begin{array}{l} 0.5 + 0.5\hat{x}, \\ 0.75 - 0.25\widehat{ixz} - 0.25\widehat{yst} + 0.25\widehat{ixyzst}, \\ 0.5 + 0.5\mu \qquad \text{if} \qquad \mu^2 = 1, \\ 0.5 + 1.3\hat{x} + 1.2\widehat{xy}. \end{array} \tag{18}$$

Most of this paper deals with the structure of idempotents, and their products, in the context of a Clifford algebra.

Since a Clifford algebra is written as a vector space, its various components can be extracted. Of these, the most important part is the real (or complex) component. We will follow the literature in extracting this component with "blade" notation:

$$\langle \alpha_1 \hat{1} + \alpha_x \hat{x} + ... \rangle_0 = \alpha_1. \tag{19}$$

If one represents a Clifford algebra in $N \times N$ matrices so that $\hat{1}$ is represented by the unit matrix, one has that $\langle \Psi \rangle_0 = \mathrm{tr}(\Psi)/N$, so this is useful for computing expectation values of operators in standard quantum mechanics. Nevertheless, we will have little use of this function.

In preference to $\langle \quad \rangle_0$ or $\mathrm{tr}(\quad)$, we will instead use squared magnitudes of Clifford algebra elements. We define $|\quad|_G^2$ for real or complex Clifford algebras as:

$$|\alpha_1 \hat{1} + \alpha_x \hat{x} + ...|_G^2 = |\alpha_1|^2 + |\alpha_x|^2 + ... \tag{20}$$

As an exercise, it may be noted that $\langle \quad \rangle_0$ and $|\quad|_G^2$ are identical for the first two idempotents listed in Eq. (18). The fourth idempotent has a component, $1.2\widehat{xy}$, that squares to a negative value and therefore the squared magnitude is greater than $0.5 = \langle 0.5 + 1.3\hat{x} + 1.2\widehat{xy} \rangle_0$. The third idempotent can go either way, depending on the nature of μ. We will be discussing this interesting fact at length in section VII where we justify the preference for using $|\quad|_G^2$ over the traditional trace.

We will sometimes call the unit vectors, $\hat{x}$, $\hat{y}$, $\hat{z}$ and $\hat{t}$, "canonical basis vectors". These elements, as well as more general products, such as $\hat{1}$, $\widehat{xy}$ or $\widehat{ixyzt}$ we will call "canonical basis elements". Vectors such as $\cos(\alpha)\hat{x} + \sin(\alpha)\hat{x})$ with α real, we will also call canonical basis vectors, but we will not refer to mixed signature unit vectors such as $\cosh(\alpha)\hat{x} + \sinh(\alpha)\hat{t}$ as canonical basis vectors. We distinguish these two types of unit vectors because the geometric squared magnitude of the mixed signature unit vectors is not unity. Similarly, mixtures of canonical basis elements that still square to unity and have unit squared magnitude, such as $\cos(\alpha)\widehat{xy} + \sin(\alpha)\widehat{xz}$ we will also call canonical basis elements. The canonical basis elements, as well as any elements that happen to square to ± 1, can conveniently be exponentiated, or multiplied by a real constant and then exponentiated. For example:

$$\begin{array}{rcl} e^{\hat{x}} & = & \cosh(1) + \sinh(1)\hat{x}, \\ e^{\alpha\hat{y}} & = & \cosh(\alpha) + \sinh(\alpha)\hat{y}, \\ e^{\alpha\hat{t}} & = & \cos(\alpha) + \sin(\alpha)\hat{t}, \\ e^{\alpha\widehat{xy}} & = & \cos(\alpha) + \sin(\alpha)\widehat{xy}, \\ e^{\alpha\widehat{xyt}} & = & \cosh(\alpha) + \sinh(\alpha)\widehat{xyt}, \\ e^{\alpha\widehat{xyzt}} & = & \cos(\alpha) + \sin(\alpha)\widehat{xyzt}, \end{array} \tag{21}$$

where α is a real (or complex) constant. One obtains the hyperbolics or regular trigonometric functions according as the canonical basis element squares to $+1$ or -1, respectively.

It is often useful to consider a dagger operation in a Clifford algebra. This operator reverses all products and takes complex conjugates of complex coefficients. For example,

$$\begin{array}{rclcrcl} (\alpha\hat{x})^\dagger & = & +\alpha^*\hat{x}, & \qquad & (\alpha\widehat{xy})^\dagger & = & -\alpha^*\widehat{xy}, \\ (\alpha\widehat{xyz})^\dagger & = & -\alpha^*\widehat{xyz}, & \qquad & (\alpha\widehat{xyzt})^\dagger & = & +\alpha^*\widehat{xyzt}, \end{array} \tag{22}$$

where α is a real or complex constant and $*$ denotes the complex conjugate.

If one is considering a complex Clifford algebra, one might choose the signature to be all positive as any negative signature elements, for example $\hat{t}$, can be replaced with $-i$ times the positive signature element $i\hat{t}$. In such case, one can see that the geometric squared magnitude can be written as

$$|M|_G^2 = \langle M^\dagger M \rangle_0. \tag{23}$$

That is, the $\dagger$ negates the portions of M that have negative signature so that the scalar part of the product is the sum of the squares of magnitudes.

Standard quantum mechanics uses squared magnitudes of spinors and we will need to compare these with the Clifford algebra squared magnitudes. In order to reduce confusion in the notation, we will write:

$$\langle A|A\rangle = ||A\rangle|_{N\times 1}^2 \tag{24}$$

for the squared magnitude of a $N \times 1$ spinor. Similarly, for the squared magnitude of a $N \times N$ matrix we will write:

$$\sum_{j=1}^{N}\sum_{k=1}^{N} |M_{jk}|^2 = |M|_{N\times N}^2. \tag{25}$$

Every now and then, to avoid confusion, we will write $|\quad|_c^2$ for the usual squared magnitude of a complex number.

We will frequently have cause to swap the order of two elements. In the usual quantum mechanics this is done by commutation or anticommutation relations. One can always move a simple canonical basis element such as $\hat{x}$, $\widehat{xyz}$ or $\widehat{ixyzt}$, from one side of an element of a Clifford algebra to the other side of the element. This is always possible because any two canonical basis elements

either commute or anticommute. The result is that the arbitrary element has some of its signs complemented. Some examples:

$$\begin{aligned}(1+\hat{x}+\hat{y}+\widehat{xy})\widehat{xyz} &= \widehat{xyz}(1+\hat{x}+\hat{y}+\widehat{xy})\\ (1-2\hat{x}+3\widehat{xt}-4\widehat{yzt})\hat{x} &= \hat{x}(1-2\hat{x}-3\widehat{xt}+4\widehat{yzt})\\ (1+\hat{x}+\hat{y}+\widehat{xy})\widehat{xyz} &= \widehat{xyz}(1+\hat{x}+\hat{y}+\widehat{xy})\\ (\cosh(\alpha)+\sinh(\alpha)\hat{x})\hat{y} &= e^{\alpha\hat{x}}\hat{y} = \hat{y}e^{-\alpha\hat{x}}\\ (\cos(\alpha)+\sin(\alpha)\widehat{xy})\widehat{xt} &= e^{\alpha\widehat{xy}}\widehat{xt} = \widehat{xt}e^{-\alpha\widehat{xy}}\end{aligned} \tag{26}$$

The first of the above follows from the fact that $\widehat{xyz}$ acts as an imaginary unit (commuting root of -1) in the Pauli algebra. The last few show the peculiar utility of this sort of operation on exponential functions.

IV. THE SCHWINGER MEASUREMENT ALGEBRA

Let a_1 be an elementary particle among some set of elementary particles $\mathcal{F}$. Let $M(a_1)$ symbolize the selective measurement that accepts particles of type a_1, and rejects all others. One can imagine some sort of Stern-Gerlach apparatus with all particles but one type being sent to a beam dump. In practice, such an apparatus may be impossible to build, for example, if we chose as an elementary particle the right handed electron. We can define the "addition" of measurements to be the less selective measurement that accepts particles of any of the included types:

$$M(a_1)+M(a_2)+M(a_3) = M(a_1+a_2+a_3). \tag{27}$$

The "multiplication" of two measurements can represent the successive application of the two measurements. Because of the physical interpretations of the symbols, addition is associative and commutative, while multiplication is at least associative. One and zero represent the trivial measurements that accept all or no particles.

Clearly, $0+M(a_1) = M(a_1)$, $1M(a_1) = M(a_1)1 = M(a_1)$, and $0M(a_1) = M(a_1)0 = 0$, so the set of measurements form an algebra. The "elementary" measurements associated with a complete set of fermions satisfy the following equations:

$$M(a_1)M(a_1) = M(a_1), \tag{28}$$

$$M(a_1)M(a_2) = 0, \quad a_1 \neq a_2, \tag{29}$$

$$\sum_{\chi} M(a_\chi) = 1 \tag{30}$$

The elementary measurements that Schwinger refers to are measurements that cannot be written as a sum of (nontrivial) measurements. The corresponding concept in mathematics is "primitive" idempotents. These are idempotents that cannot be written as the sum of (nontrivial) idempotents. Our use of the Schwinger measurement algebra will be in its method of defining the elementary particles according to the spectrum of primitive idempotents of Clifford algebras.

In the density matrix formalism, a "pure" density matrix is one that satisfies Eq. (28) and is idempotent. There are advantages to using a density matrix approach to the formalism of quantum mechanics. Brown and Hiley write:

> In the usual approach to quantum mechanics, the density operator is, unfortunately, not introduced as a primitive notion in the theory. Rather, it is introduced almost as an after-thought when it is found necessary to deal with mixed states. But using the density operator as a starting point has the advantage of including both pure states and mixed states together and of satisfying the idempotent condition $\rho = \rho^2$.[1]

In fact, the SMA and the density matrix formalisms are compatible as we now show. Since we are treating particles with different handedness as distinct particles, we can compare the two algebras when the states are distinct. Let $\iota_A = |+z\rangle\langle+z|$ be the pure density matrix representation of a fermion with spin $+1/2$ in the $\hat{z}$ direction and let $\iota_B = |-z\rangle\langle-z|$ be the oppositely directed state. Then the product of these two density matrices is zero:

$$\begin{aligned}\iota_A\,\iota_B &= |+z\rangle\langle+z||-z\rangle\langle-z|,\\ &= |+z\rangle(\langle+z|-z\rangle)\langle-z| = 0.\end{aligned} \tag{31}$$

This is identical to the Schwinger measurement algebra rule Eq. (29). In addition, the sums over primitive elements of each algebra are unity, for example:

$$\begin{aligned}|+z\rangle\langle+z|+|-z\rangle\langle-z| &= (1,0)^\dagger(1,0)+(0,1)^\dagger(0,1)\\ &= \begin{pmatrix}1&0\\0&1\end{pmatrix}\end{aligned} \tag{32}$$

Thus the Schwinger measurement algebra is a generalization of the density matrix representation, in that it allows the inclusion of arbitrarily distinct particles.

V. GENERAL MEASUREMENTS AND ORIENTATION

In addition to the measurement symbols defined in the previous section, Schwinger defines "general measurement symbols" that model a measurement where the measurement modifies the particle. For an incoming state of b' and an outgoing state of a', he writes $M(a',b')$. These general measurements can be chained together. For example:

$$M(a',d') = M(a',b')M(b',c')M(c',d') \tag{33}$$

We will not be using these measurements because they produce an unphysical degree of freedom (or gauge) as we will now show in this section.

Consider the product of two measurement's of Schwinger's general type:

$$M(a,b)M(c,d) = \langle b|c\rangle\ M(a,d). \tag{34}$$

As long we assume that $\langle b|c\rangle$ is a complex number, it makes mathematical sense to factor the general measurements. For example, if we factor $M(a,b)$ and $M(c,d)$ into bras and kets:

$$M(a,b) = |a\rangle\langle b|, \qquad M(c,d) = |c\rangle\langle d|, \tag{35}$$

then Eq. (34) follows as a result of the fact that the complex number $\langle b||c\rangle$ commutes with $|a\rangle$.

Schwinger's justification for assuming complex numbers in the SMA is given by two passages, the first of which is as follows:

> The examples of compound measurements that we have already considered involve the passage of all systems or no systems between the two stages, as represented by the multiplicative numbers 1 and 0. More generally, measurements of properties B, performed on a system in a state c' that refers to properties incompatible with B, will yield a statistical distribution of possible values. Hence, only a determinate fraction of the systems emerging from the first stage will be accepted by the second stage. We express this by the general multiplication law
>
> (1.14) $M(a',b')M(c',d') = \langle b'|c'\rangle M(a',d')$,
>
> where $\langle b'|c'\rangle$ is a number characterizing the statistical relation between the states b' and c'. In particular,
>
> (1.15) $\langle a'|a''\rangle = \delta(a',a'')$.[5]

Of course probabilities are given by $P = |\langle b'|c'\rangle|^2$. The second half of Schwinger's argument for complex numbers[2] is the following (in which I replace symbols of the form $M(a)$ with $M(a,a)$ for clarity):

> 1.6 STATISTICAL INTERPRETATION
>
> It should be observed that the general multiplication law and the definition of the trace are preserved if we make the substitutions
>
> 1.34) $M(a',b') \to \lambda(a')^{-1}\ M(a',b')\ \lambda(b')$
> $\langle a'|b'\rangle \to \lambda(a')\langle a'|b'\rangle\lambda(b')^{-1}$,
>
> where the numbers $\lambda(a')$ and $\lambda(b')$ can be given arbitrary non-zero values. The elementary measurement symbols $M(a')$ and the transformation function $\langle a'|a''\rangle$ are left unaltered. In view of this arbitrariness, a transformation function $\langle a'|b'\rangle$ cannot, of itself, possess a direct physical interpretation but must enter in some combination that remains invariant under the substitution (1.34).
>
> The appropriate basis for the statistical interpretation of the transformation function can be inferred by a consideration of the sequence of selective measurements $M(b',b')M(a',a')M(b',b')$, which differs from $M(b',b')$ in virtue of the disturbance attendant upon the intermediate A-measurement. Only a fraction of the systems selected in the initial B-measurement is transmitted through the complete apparatus. Correspondingly, we have the symbolic equation
>
> (1.35) $M(b',b')M(a',a')M(b',b') = p(a',b')M(b',b')$,
>
> where the number
>
> (1.36) $p(a',b') = \langle a'|b'\rangle\langle b'|a'\rangle$
>
> is invariant under the transformation (1.34). If we perform an A-measurement that does not distinguish between two (or more) states, there is a related additivity of the numbers $p(a',b')$,
>
> (1.37) $M(b',b')\,(M(a',a') + M(a'',a''))\,M(b',b')$
> $= (p(a',b') + p(a'',b'))\,M(b',b')$,
>
> and, for the A-measurement that does not distinguish among any of the states, there appears
>
> (1.38) $M(b',b')\,(\sum_{a'} M(a',a'))\,M(b',b') = M(b',b')$,
>
> whence
>
> (1.39) $\sum_{a'} p(a',b') = 1$.
>
> These properties qualify $p(a',b')$ for the role of the probability that one observes the state a' in a measurement performed on a system known to be in the state b'. But a probability is a real, non-negative number. Hence we shall impose an admissible restriction on the numbers appearing in the measurement algebra, by requiring that $\langle a'|b'\rangle$ and $\langle b'|a'\rangle$ form a pair of complex conjugate numbers
>
> (1.40) $\langle b'|a'\rangle = \langle a'|b'\rangle^*$,
>
> for then
>
> (1.41) $p(a',b') = |\langle a'|b'\rangle|^2 \geq 0$.
>
> To maintain the complex conjugate relation (1.40), the numbers $\lambda(a')$ of (1.34) must obey
>
> (1.42) $\lambda(a')^* = \lambda(a')^{-1}$,

[2] In 1997, L. P. Horwitz[10] showed that the complex numbers in the SMA could be replaced by quaternions.

and therefore have the form

(1.43) $\lambda(a') = e^{i\phi(a')}$

in which the phases $\phi(a')$ can assume arbitrary real values.[5]

The above argument has difficulty upon extending it to situations with more than one intervening measurement. In fact, just such situations is the topic of much of this paper. In such a situation, following Schwinger's notation, one would replace $p(a', b')$ with $p(a', c', b')$ defined as:

$$\begin{aligned} &M(b', b')M(a', a')M(c', c')M(b', b') \\ &= p(a', c', b')M(b', b'). \end{aligned} \tag{36}$$

From the example of spin$-1/2$ in the x, y and z directions, $M(b', b') = (1+\sigma_z)/2$, $M(a', a') = (1+\sigma_x)/2$, and $M(c', c') = (1+\sigma_y)/2$, we know that, $p(a', c', b')$ cannot in general be a real number, but instead must be complex. In this case, we know that the correct probability that must be associated with the four consecutive measurements is not $p(a', c', b')$, but instead is $|p(a', c', b')|$.

There are two effects going on here. First, the incompatibility of the sequence of internal measurements decreases the probability of a particle making it through the sequence of measurements. Second, the sequence changes the phase of the particle. We can attempt to split these two effects by using Schwinger's general measurements, i.e. measurements of form $M(a', b')$, but by so doing we run into a difficulty. The underlying problem is that the measurements of the form $M(a', b')$ change a particle from one sort to another without specifying any sort of phase change.

For example, let us naively suppose that

$$M(a', a')M(b', b') = \sqrt{p(a', b')}\; M(a', b'), \tag{37}$$

and that $p(a', b') = p(b', a')$. Then

$$\begin{aligned} &M(b', b')M(a', a')M(b', b') \\ &= M(b', b')M(a', a')M(a', a')M(b', b') \\ &= \sqrt{p(a', b')}M(b', a')\sqrt{p(b', a')}M(a', b') \\ &= p(a', b')M(b', a')M(a', b') \\ &= p(a', b')M(b', b'), \end{aligned} \tag{38}$$

and Schwinger's equation (1.35) is satisfied. However, such a substitution would give

$$\begin{aligned} &M(b', b')M(a', a')M(c', c')M(b', b') \\ &= M(b', b')M(a', a')\; M(a', a')M(c', c')\; M(c', c')M(b', b') \\ &= \sqrt{p(b', a')}\sqrt{p(a', c')}\sqrt{p(c', b')}\; M(b', a')\; M(a', c')\; M(c', b') \\ &= \sqrt{p(b', a')}\sqrt{p(a', c')}\sqrt{p(c', b')}\; M(b', b'). \end{aligned} \tag{39}$$

This gives the correct amplitude for $p(a', c', b')$, but its phase is always zero which is incorrect.

So if we are to write $M(a', b')$ as a multiple of $M(a', a')M(b', b')$ we have to adjust our substitution to something of the form $\sqrt{p(a', b')}e^{+i\theta(a', b')}\; M(a', b') = M(a', a')M(b', b')$ with $p(a', b') = p(b', a')$ and $\theta(a', b') = -\theta(b', a')$. Using this in the calculation of Eq. (39) gives:

$$\begin{aligned} &M(b', b')M(a', a')M(c', c')M(b', b') \\ &= e^{i(\theta(b', a')+i\theta(a', c')+i\theta(c', b'))}p(a', c', b')M(b', b'). \end{aligned} \tag{40}$$

For the example of the Pauli algebra, the desired phase is half the area of the spherical triangle defined by the three vectors a', b' and c'. A solution can therefore be obtained by choosing a an arbitrary unit vector, for example $z = (0, 0, 1)$ and defining θ as

$$\theta(a', b') = \; S(a', b', z)/2 \tag{41}$$

where S is the (oriented) area of the spherical triangle defined by the given unit vectors.[3]

The definition of θ given by Eq. (41) is unsatisfactory in that it amounts to choosing a preferred orientation, $(0, 0, 1)$. It is easily seen that there are no symmetrical solutions to the problem of defining θ. Since we would prefer to keep our mathematics as symmetric as possible, we therefore avoid the use of Schwinger's general measurement symbols $M(a', b')$ as their definition in terms of products of the measurement symbols $M(a')$ requires an unphysical selection of a preferred orientation.

In Schwinger's measurement algebra, the connection to the annihilation and creation operators of the usual quantum field theory is obtained by introducing a "fictitious" null, or vacuum state 0 as one of the states in the more general measurement $M(a, b)$. From his book:

> The uncontrollable disturbance attendant upon a measurement implies that the act of measurement is indivisible. That is to say, any attempt to trace the history of a system during a measurement process usually traces the nature of the measurement that is being performed. Hence, to conceive of a given selective measurement $M(a', b')$ as a compound measurement is without physical implication. It is only of significance that the first stage selects systems in the state b', and that the last one produces them in the state a'; the interposed states are without meaning for the measurement as a whole. Indeed, we can even invent a non-physical state to serve as the intermediary. We shall call this mental construct the null state 0, and write
>
> (2.1) $M(a', b') = M(a', 0)M(0, b')$.
>
> The measurement that selects a system in the state b' and produces it in the null state,[4]
>
> (2.2) $M(0, b') = \psi(b')$,

[3] This fact is proven below.

[4] Schwinger writes his annihilation and creation operators as $\Phi(b')$ and $\Psi(a')$. In the interest of conforming to modern practice, I've replaced these with $\psi(b')$ and $\psi^\dagger(a')$, respectively.

can be described as the annihilation of a system in the state b'; and the production of a system in the state a' following its selection from the null state,

(2.3) $(a', 0) = \psi^\dagger(a')$,

can be characterized as the creation of a system in the state a'. Thus the content of (2.1) is the indiscernibility of $M(a', b')$ from the compound process of the annihilation of a system in the state b' followed by the creation of a system in the state a',

(2.4) $M(a', b') = \psi^\dagger(a')\psi(b')$ [5]

When Schwinger reprinted the above referenced book in 1991, he wrote a special preface with the comment: "Instead of the symbol of measurement: $M(a', b')$, I now write $|a'b'|$, combining reference to what is selected and what is produced, with an indication that the act of measurement has a beginning and an end. Then, with the conceptual analysis of $|a'b'|$ into two stages, one of annihilation and one of creation, as symbolized by $|a'b'| = |a'\rangle\langle b'|$, the fictitious null state, and the symbols Ψ and Φ can be discarded." At that time, the use of spontaneous symmetry breaking of the vacuum was already well established for the standard model. One wonders what he thought of it.

In avoiding Schwinger's general measurement symbols we also must avoid the vacuum. But in so doing, we obtain relief from the otherwise unavoidable unphysical gauge freedoms. Analogously, a similar problem in the Dirac equation disappears when it is converted to density matrix form, see Section II.

VI. SPINORS, SQUARE AND GEOMETRIC

In rejecting Schwinger's general measurement symbols, we also must reject Schwinger's use of complex numbers. This presents difficulties in providing a probability interpretation to our algebra. In moving from the usual probability interpretation of spinors to a new method of deriving probabilities in an entirely geometric way (that has no unphysical degrees of freedom), it will be useful to use the concept of "square spinors"–which we review in this section.

We will use the example of Dirac spinors. The Dirac equation has operators typically consisting of 4×4 matrices, and states consisting of 4×1 vectors. Writing the equation with the dimensions of the objects explicitly included as a suffix:

$$(\gamma^\mu \partial_\mu)_{4\times4}\ \psi_{4\times1} = m\psi_{4\times1}. \tag{42}$$

By the laws of matrix multiplication, we can replace the $\psi_{4\times1}$ vector with a 4×4 matrix. We simply put the vector into one of the columns of the matrix and keep the other columns zero. For example, the vector:

$$\psi_{4\times1} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}, \tag{43}$$

can be replaced by the matrix:

$$\psi_{4\times1} = \begin{pmatrix} 0 & \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & \alpha_3 & 0 & 0 \\ 0 & \alpha_4 & 0 & 0 \end{pmatrix}, \tag{44}$$

giving a matrix Dirac equation in every way the same as the vector equation:

$$(\gamma^\mu \partial_\mu)_{4\times4}\ \psi_{4\times4} = m\psi_{4\times4}, \tag{45}$$

except that it now has three columns of unneeded zeros.

An obvious interpretation of the extra three columns is that they correspond to distinct particles. This provides a natural explanation for how nature arranges to share the same propagator among several different (and therefore non interfering) particles such as electrons and quarks. The 4×4 matrices of $\mathcal{CL}(4, 1)$ are big enough to fit, for example, the electron and the three colors of up quark only. This is just half the number of degrees of freedom required for a single generation of elementary fermions. Greg Trayling puts the 8 elementary fermions of a single generation into the eight columns[4, Table 2.] [11] of the real 8×8 matrices representing $\mathcal{CL}(7, 0)$, assuming 4 hidden spatial dimensions.

The complex 4×4 matrices can be written as a real vector space over products of the gamma matrices $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4\} \equiv \{\hat{t}, \hat{x}, \hat{y}, \hat{z}, \hat{s}\}$ and this gives a completely real geometric interpretation of any 4×4 complex matrix. Similarly, the complex 2×2 matrices can be written as a real vector space over products of Pauli matrices $\{\sigma_x, \sigma_y, \sigma_z\} \equiv \{\hat{x}, \hat{y}, \hat{z}\}$. The interpretation depends on the choice of representation. So using square spinors can give us a geometric interpretation of both our operators and our states.

The geometric interpretation is important and we will illustrate it using the Weyl (or chiral) representation of the Dirac matrices.[12, §3.2] For brevity, we will work in the more usual complex representation of $\ni, \infty$ rather than the real representation of $\triangle, \infty$. The basis set for the 4×4 complex matrices is given by the various products of $\hat{x}, \hat{y}, \hat{z}$ and $\hat{t}$. Using $\widehat{i\hat{y}}$ instead of $\hat{y}$ puts the matrices into real values which will be convenient. Written with the matrix representation under the corresponding

geometric form, the sixteen products are:

$$
\begin{array}{cccc}
\hat{1} & \widehat{ixy} & \widehat{zt} & \widehat{ixyzt} \\
\left|\begin{array}{cccc} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right| &
\left|\begin{array}{cccc} 1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{array}\right| &
\left|\begin{array}{cccc} 1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{array}\right| &
\left|\begin{array}{cccc} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{array}\right| \\
\widehat{yzt} & \hat{x} & \hat{iy} & \widehat{xzt} \\
\left|\begin{array}{cccc} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{array}\right| &
\left|\begin{array}{cccc} 0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{array}\right| &
\left|\begin{array}{cccc} 0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{array}\right| &
\left|\begin{array}{cccc} 0&0&0&1\\0&0&-1&0\\0&1&0&0\\-1&0&0&0\end{array}\right| \\
\hat{it} & \hat{z} & \widehat{ixyt} & \widehat{ixyz} \\
\left|\begin{array}{cccc} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{array}\right| &
\left|\begin{array}{cccc} 0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{array}\right| &
\left|\begin{array}{cccc} 0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{array}\right| &
\left|\begin{array}{cccc} 0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{array}\right| \\
\widehat{iyz} & \widehat{xz} & \widehat{iyt} & \hat{xt} \\
\left|\begin{array}{cccc} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{array}\right| &
\left|\begin{array}{cccc} 0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{array}\right| &
\left|\begin{array}{cccc} 0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0\end{array}\right| &
\left|\begin{array}{cccc} 0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{array}\right|
\end{array}
\tag{46}
$$

The above 16 basis elements are organized in four rows with each row sharing the same four nonzero matrix elements. For example, $\widehat{iyz}$, $\widehat{xz}$, $\widehat{iyt}$ and $\widehat{xt}$ correspond to the only matrices that are nonzero in positions $(1,2)$, $(2,1)$, $(3,4)$ and $(4,3)$. Thus it is easy to invert the basis and to write an arbitrary complex 4×4 matrix A, in geometric form:

$$\alpha(A) = \alpha_1\hat{1} + \alpha_x\hat{x} + ... + \alpha_{xyzt}\widehat{xyzt}. \tag{47}$$

Reading off the 16 basis matrices, we have:

$$
\begin{aligned}
\alpha_1 &= (A_{11}+A_{22}+A_{33}+A_{44})/4 \\
\alpha_{xy} &= (A_{11}-A_{22}+A_{33}-A_{44})/4i \\
\alpha_{zt} &= (A_{11}-A_{22}-A_{33}+A_{44})/4 \\
\alpha_{xyzt} &= (A_{11}+A_{22}-A_{33}-A_{44})/4i \\
\alpha_{yzt} &= (A_{14}+A_{23}+A_{32}+A_{41})/4i \\
\alpha_x &= (A_{14}+A_{23}-A_{32}-A_{41})/4 \\
\alpha_{iy} &= (A_{14}-A_{23}-A_{32}+A_{41})/4i \\
\alpha_{xzt} &= (A_{14}-A_{23}+A_{32}-A_{41})/4 \\
\alpha_t &= (A_{13}+A_{24}+A_{31}+A_{42})/4 \\
\alpha_z &= (A_{13}-A_{24}-A_{31}+A_{42})/4 \\
\alpha_{xyt} &= (A_{13}-A_{24}+A_{31}-A_{42})/4i \\
\alpha_{xyzt} &= (A_{13}+A_{24}-A_{31}-A_{42})/4i \\
\alpha_{yz} &= (A_{12}+A_{21}+A_{34}+A_{43})/4i \\
\alpha_{xz} &= (A_{12}-A_{21}+A_{34}-A_{43})/4 \\
\alpha_{yt} &= (A_{12}-A_{21}-A_{34}+A_{43})/4i \\
\alpha_{xt} &= (A_{12}+A_{21}-A_{34}-A_{43})/4
\end{aligned}
\tag{48}
$$

If one multiplies the first four of the above equations by their complex conjugates and adds, one discovers that the cross terms cancel and:

$$
\begin{aligned}
&|\alpha_1|^2 + |\alpha_{xy}|^2 + |\alpha_{zt}|^2 + |\alpha_{xyzt}|^2 \\
&\quad = (|A_{11}|^2 + |A_{22}|^2 + |A_{33}|^2 + |A_{44}|^2)/4.
\end{aligned}
\tag{49}
$$

A similar coincidence applies to the remaining 12 equations. Thus we have that the geometric squared magnitude of Eq. (20) is related to the matrix squared magnitude. That is,

$$4|\alpha(A)|^2 = |A|^2_{4\times 4} = \sum_j \sum_k |A_{jk}|^2. \tag{50}$$

This is not generally true, but depends on the representation, as we will discuss in the next section.

Eq. (48) gives the conversion for taking a complex 4×4 matrix A to the complex Clifford algebra $\mathcal{CL}(3,1)$. To convert to the real Clifford algebra $\mathcal{CL}(4,1)$ one uses $i = \widehat{xyzst}$ and expands each line into a real and complex part. The first few conversions give:

$$
\begin{aligned}
\alpha_1 &= +\mathcal{R}(A_{11}+A_{22}+A_{33}+A_{44})/4 \\
\alpha_{xyzst} &= +\mathcal{I}(A_{11}+A_{22}+A_{33}+A_{44})/4 \\
\alpha_{zst} &= -\mathcal{R}(A_{11}-A_{22}+A_{33}-A_{44})/4 \\
\alpha_{xy} &= +\mathcal{I}(A_{11}-A_{22}+A_{33}-A_{44})/4 \\
\alpha_{zt} &= +\mathcal{R}(A_{11}-A_{22}-A_{33}+A_{44})/4 \\
\alpha_{xys} &= -\mathcal{I}(A_{11}-A_{22}-A_{33}+A_{44})/4 \\
\alpha_s &= +\mathcal{R}(A_{11}+A_{22}-A_{33}-A_{44})/4 \\
\alpha_{xyzt} &= +\mathcal{I}(A_{11}+A_{22}-A_{33}-A_{44})/4 \\
&...
\end{aligned}
\tag{51}
$$

Given an $N\times N$ matrix representation of a Clifford algebra, a set of important matrices are the diagonal primitive idempotents. These are matrices that are zero but for a single one on the diagonal. For the Pauli algebra, the two diagonal primitive idempotents are:

$$\iota_\pm = \frac{1\pm\sigma_z}{2}. \tag{52}$$

The four diagonal primitive idempotents of the Weyl representation of the Dirac algebra are:

$$\iota_{\pm\pm} = \frac{1\pm\widehat{ixy}}{2}\frac{1\pm\hat{zt}}{2}. \tag{53}$$

where the $\pm$ are to be taken independently.

Another important matrix in a representation is the "democratic" matrix which has all elements equal to $1/N$:

$$D_{N\times N} = \frac{1}{N}\begin{pmatrix} 1 & 1 & \\ 1 & 1 & \\ & & ... \end{pmatrix}. \tag{54}$$

Note that the democratic matrix is a primitive idempotent. For the example of the Pauli spin matrices,

$$D_{2\times 2} \equiv (\hat{1}+\sigma_x)/2. \tag{55}$$

For the Dirac matrices in the Weyl representation,

$$
\begin{aligned}
D_{4\times 4} &\equiv (\hat{1} - \widehat{yzt} + \hat{it} + \widehat{iyz})/4 \\
&\equiv (1+\hat{it})(1+\widehat{iyz})/4.
\end{aligned}
\tag{56}
$$

The N diagonal primitive idempotents, together with the democratic matrix, fully determine the representation. In fact, the product

$$B_{jk} = \iota_j \; D_{N\times N} \; \iota_k \tag{57}$$

is the matrix with 1 in position (j,k) and all other matrix positions zero.

VII. PROBABILITIES AND THE SMA

The logic behind Schwinger's statistical interpretation of the measurement symbols requires factoring into bras and kets. Having rejected this, his statistical interpretation also becomes suspect. Since the elements of the SMA are all Stern-Gerlach filters, our only need for probabilities arises in the intensities of beams. For this, we need a probability that is proportional to the square of the amplitude of the beam. In this section we derive a probability interpretation based on the Clifford algebraic squared magnitude defined in Eq. (20). Thus instead of using traces to extract real numbers from operators, we will be using the squared magnitude typically used to extract probabilities from spinors, but at the same time, instead of using spinors to represent states, we will use operators.

In the spinor representation of QM, there is a conflict between normalization and linearity.[5] The linear combination of two normalized spinor wave functions is generally not a normalized spinor wave function. Requiring the spinor theory to be closed under linear combinations therefore requires that we include normalization factors in our calculations. For example, when computing a probability of a transition between two unnormalized spinor states A and B we could use:

$$P_{AB} = \frac{\langle A|B\rangle\langle B|A\rangle}{\langle A|A\rangle\langle B|B\rangle}. \tag{58}$$

On the other hand, if we give up the notion of linearity and arrange for our spinor wave functions to be normalized, then we can calculate probabilities in a simpler way:

$$P_{AB} = \langle A|B\rangle\langle B|A\rangle. \tag{59}$$

With density matrices, we have no easy way to retain a form of linearity, so there is no reason to sacrifice normality for linearity. In addition, idempotent normalization is unique; they do not possess the arbitrary complex phases that spinors do.

[5] In contrast to classical E&M, quantum mechanics, even in the usual spinor formalism, is not physically linear. Three times a spinor wave function is a wave function that corresponds to the same physical situation (with the normalization changed), not a physical situation with three times as many particles or particles that are three times stronger.

Since we are assuming that the elementary particles each has its own primitive measurement, it's natural to represent the elementary particles with those measurements. Unfortunately, this is too restrictive because we can produce measurements that are different from the elementary ones but that nevertheless, correspond to physical experiments that pass only the given particles.

For example, consider a sequence of four Stern-Gerlach filters oriented in the z, y, x and z direction, $M(zyxz)$. Since the beginning and ending filters are both oriented in the z direction, this will be equivalent to a single Stern-Gerlach filter oriented in the z direction, but with an amplitude of $\sqrt{1/8}$ and a phase change of $\pi/4$. The amplitude gives a probability of 1/8, as we expect from forcing the particle through three consecutive completely incompatible spin$-1/2$ measurements. The compound measurement is represented in the SMA by:

$$\begin{aligned}
&M(zyxz) = (1+\hat{z})(1+\hat{x})(1+\hat{y})(1+\hat{z})/16 \text{ or}\\
&16M(zyxz) = (1+\hat{z}+\hat{x}+\hat{y}+\hat{z}+\hat{z}\hat{z}+\hat{x}\hat{y}\\
&\quad +\hat{z}\hat{x}\hat{y}+\hat{z}\hat{x}\hat{z}+\hat{z}\hat{y}\hat{z}+\hat{x}\hat{y}\hat{z}+\hat{z}\hat{x}\hat{y}\hat{z}\\
&= 2+2\hat{z}+\hat{x}+\hat{y}+2\hat{x}\hat{y}-\hat{x}-\hat{y}+2\hat{x}\hat{y}\hat{z}\\
&= 2(1+\hat{z})(1+\hat{x}\hat{y}\hat{z}) \text{ so}\\
&M(zyxz) = M(z)(1+\widehat{xyz})/4\\
&= \tfrac{1}{\sqrt{8}}\, e^{\frac{\pi}{4}\widehat{xyz}}\, M(z).
\end{aligned} \tag{60}$$

Thus the difference between $M(z)$ and $M(zyxz)$ is that the amplitude is smaller by a factor of 8 and, there has been a geometric phase shift of $\pi/4$ $\widehat{xyz}$. In $\mathcal{CL}(3,0)$, $\widehat{xyz}$ is a geometric imaginary unit, so for this Clifford algebra we could treat the $\pi/4$ as a complex phase, but this does not hold true for arbitrary Clifford algebras. For example, $\mathcal{CL}(3,1)$ doesn't possess a geometric imaginary unit. So, in general, a given elementary particle will have to be represented by a set of distinct Clifford algebraic numbers.

Physically, the phase need not concern us if we have another method of obtaining the probability. And the Clifford algebra can take care of the phases by itself. So we need not use complex numbers if we can obtain the probabilities direct from the Clifford algebra. Accordingly, let us reexamine $M(zyxz)$:

$$\begin{aligned}
M(zyxz) &= M(z)(1+\widehat{xyz})/4\\
&= (1+\hat{z}+\widehat{xyz}+\widehat{xy})/8, \quad \text{so}\\
|M(zyxz)|^2_G &= 4/64 = 1/16.
\end{aligned} \tag{61}$$

On the other hand,

$$|M(z)|^2_G = |(1+\hat{z})/2|^2 = 1/4+1/4 = 1/2, \tag{62}$$

so we see that the square of the beam amplitude, or probability, is given by $P(zyxz) = |M(zyxz)|^2_G/|M(z)|^2_G$. This suggests that we assume that probabilities are proportional to squared magnitudes. That $|M(z)|^2_G = 1/2$ we can interpret as the fact that just half of an unpolarized beam of spin$-1/2$ particles will survive a single Stern-Gerlach filter.

As a test, let us consider a sequence of two Stern-Gerlach filters oriented with the first one in the v direction and the second in the u direction. We know that

the beam amplitude will be reduced by the square root of $P(u,v) = (1+\cos(\theta))/2$ where θ is the angle between u and v:

$$\begin{aligned}
&|M(uv)|^2_G = |(1+\hat{u})/2\ (1+\hat{v})/2)|^2_G \\
&= |1+\hat{u}+\hat{v}+\hat{u}\hat{v}|^2_G/16 \\
&= |1+u_xv_x+u_yv_y+u_zv_z+(u_x+v_x)\hat{x}+(u_y+v_y)\hat{y} \\
&\quad +(u_z+v_z)\hat{z}+(u_xv_y-u_yv_x)\widehat{xy} \\
&\quad +(u_xv_z-u_zv_x)\widehat{xz}+(u_yv_z-u_zv_y)\widehat{yz}|^2_G/16 \\
&= ((1+u\cdot v)^2+2+2u\cdot v+|u\times v|^2)/16 \\
&= (1+2\cos(\theta)+\cos^2(\theta)+2+2\cos(\theta)+\sin^2(\theta))/16 \\
&= (4+4\cos(\theta))/16 = (1+\cos(\theta))/4.
\end{aligned} \tag{63}$$

Thus $|M(uv)|^2_G/|M(u)|^2_G$ gives the correct probability in this case as well.

More generally, let ψ be a normalized $N\times 1$ spinor (so that $(\psi\psi^\dagger)^2 = \psi\psi^\dagger$), and M be an $N\times N$ matrix (over the complex numbers) that defines a series of operations (i.e. a complicated filter) to be performed on the state ψ. Then the probability of a particle surviving the sequence of filters is given by:

$$\begin{aligned}
P(M,\psi) &= |M\psi|^2_{N\times 1}\,, \\
&= \psi^\dagger M^\dagger M\psi, \\
&= \mathrm{tr}(M\psi\psi^\dagger M^\dagger), \\
&= \mathrm{tr}(M\psi\psi^\dagger\psi\psi^\dagger M^\dagger), \\
&= \mathrm{tr}(\textstyle\sum_n |n\rangle\langle n|M\psi\psi^\dagger \sum_m |m\rangle\langle m|\psi\psi^\dagger M^\dagger),
\end{aligned} \tag{64}$$

where $|m\rangle$ is the spinor with a 1 in its mth position and the rest zero. Continuing:

$$\begin{aligned}
P(M,\psi) &= \textstyle\sum_m\sum_n \mathrm{tr}(|n\rangle\langle n|M\psi\psi^\dagger|m\rangle\langle m|\psi\psi^\dagger M^\dagger), \\
&= \textstyle\sum_m\sum_n \mathrm{tr}(\langle n|M\psi\psi^\dagger|m\rangle\langle m|\psi\psi^\dagger M^\dagger|n\rangle), \\
&= \textstyle\sum_m\sum_n (\langle n|M\psi\psi^\dagger|m\rangle)(\langle m|\psi\psi^\dagger M^\dagger|n\rangle), \\
&= \textstyle\sum_m\sum_n \left|\langle n|M\psi\psi^\dagger|m\rangle\right|^2_c\,, \\
&= |M\psi\psi^\dagger|^2_{N\times N},
\end{aligned} \tag{65}$$

and we have that the probability of ψ surviving the filter M is given by the squared magnitude of the product of the matrix M with the density matrix of ψ. For a generalization of this principle, see [13].

Thus the usual probability formula for density matrices $\mathrm{tr}(M\rho)$, gives the same predictions as the squared matrix magnitude $|M\rho|^2_{N\times N}$. It remains to relate these values with the natural squared magnitude of a Clifford algebra. Reviewing the derivation of Eq. (50), we see that the relationship will be determined by how the matrix elements are defined in terms of the geometry.

As noted in the previous section, the N diagonal primitive idempotents, together with the democratic matrix, fully determine the relationship between the matrix representation and the Clifford algebra. In matrix form, these $N+1$ elements are particularly simple, and as seen in Eq. (57) they provide a natural way of computing $|\quad|^2_{N\times N}$ in geometric form. But the number of ways that this can be done depends on how these $N+1$ primitive idempotents can be chosen and to understand that, we need to know a little about how primitive idempotents appear in a Clifford algebra.

The subject of the primitive idempotents of a Clifford algebra are complicated with the appearance of Radon-Hurwitz numbers[6, §17.5] but we need not understand the subject to this depth. Instead, let us take as an example a Clifford algebra that has two non trivial commuting square roots of unity. Such an algebra would physically correspond to a finite quantum system with two independent quantum numbers (such as the spin and particle type of the Dirac algebra).

Let e_1 and e_2 be commuting non scalar roots of unity. That is,

$$(e_1)^2 = 1, \qquad (e_2)^2 = 1, \qquad e_1e_2 = e_2e_1, \tag{66}$$

and suppose that there are no other non scalar roots of unity. Then it turns out that the $2^2 = 4$ elements of the algebra defined by taking independently the signs in

$$\iota_{\pm\pm} = \frac{1\pm e_1}{2}\frac{1\pm e_2}{2} \tag{67}$$

are a mutually annihilating complete set of primitive idempotents. That is, the four elements are each primitive idempotents, they sum to 1, the products of two different of them are zero, and none of them can be written as a nontrivial sum of two idempotents. Any projection operator can be written as a (possibly trivial) sum of elements of some such set of primitive idempotents. For the Weyl representation of the Dirac equation given by Eq. (46), the diagonal primitive idempotents are defined geometrically by Eq. (67) with $e_1 = \widehat{ixy}$ and $e_2 = \widehat{zt}$ as is most easily seen by comparing Eq. (53) to the above, while the democratic primitive idempotent is obtained with commuting roots of unity given by $d_1 = \widehat{it}$, $d_2 = \widehat{iyz}$. Note that these two sets of commuting roots of unity share no common elements.

Given a representation of a Clifford algebra, we can always extract the e_χ and d_χ sets of commuting roots of unity. The reverse problem, how to choose two commuting sets of unity is more difficult. We now illustrate how this can be done by looking at the Dirac algebra. The Clifford algebra element that gives the democratic primitive idempotent is:

$$\iota_D = \frac{1+d_1}{2}\frac{1+d_2}{2}. \tag{68}$$

The four diagonal primitive idempotents are:

$$\begin{aligned}
\iota_1 &= \tfrac{1-e_1}{2}\tfrac{1-e_2}{2}, \\
\iota_2 &= \tfrac{1+e_1}{2}\tfrac{1-e_2}{2}, \\
\iota_3 &= \tfrac{1-e_1}{2}\tfrac{1+e_2}{2}, \\
\iota_4 &= \tfrac{1+e_1}{2}\tfrac{1+e_2}{2}.
\end{aligned} \tag{69}$$

And the matrix of zeroes except for a one at position (j,k) is given by:

$$M_{jk} = N\iota_j\ \iota_D\ \iota_k, \tag{70}$$

where $N = 2^n = 4$. To have this set of matrices be a representation of the Clifford algebra, we need to have

$$M_{jk}M_{lm} = \delta^l_k\ M_{jm} \tag{71}$$

map faithfully into the Clifford algebra:

$$N^2(\iota_j\ \iota_D\ \iota_k)\ (\iota_l\ \iota_D\ \iota_m) = \delta^l_k\ N(\iota_j\ \iota_D\ \iota_m). \tag{72}$$

The fact that $\iota_k\ \iota_l = \delta^l_k \iota_k$ shows that this is equivalent to requiring:

$$N^2 \iota_j\ \iota_D\ \iota_k\ \iota_D\ \iota_m = N \iota_j\ \iota_D\ \iota_m. \tag{73}$$

Writing $1 = \sum_j \iota_j = \sum_m \iota_m$ shows that the above is equivalent to:

$$N \iota_D\ \iota_k\ \iota_D = \iota_D. \tag{74}$$

Since $\sum_{k=1}^N \iota_k = 1$, the above equation is automatically true in average. But it also shows that we can't choose the d_χ equal to the e_χ as that would result in either 0 or $N\iota_D$ on the left hand side. But no matter how ι_k is chosen, the two sides of the above equation must be at least proportional if not equal as both sides act identically as a multiple of the primitive projection operator ι_D.

We can therefore take the trace to determine if the constant of proportionality in Eq. (74) is correct:

$$\begin{aligned} \mathrm{tr}(N\iota_D\ \iota_k\ \iota_D) &= N\mathrm{tr}(\iota_D\ \iota_D\ \iota_k) \\ &= N\mathrm{tr}(\iota_D\ \iota_k). \end{aligned} \tag{75}$$

We expect that this is equal to $\mathrm{tr}(\iota_D) = 1/N$ because the only compatible degrees of freedom between ι_D and ι_k is presumed to be the scalar, and both these primitive idempotents have scalar part equal to $1/N$. However, there is a detail that needs to be mentioned.

Counting up the number of degrees of freedom, we see that the number of distinct matrices M_{jk} is apparently $N^2 = 4^n$, and in order for this to be equal to the number of degrees of freedom in the Clifford algebra $\mathcal{CL}(p,g)$, we would have to have that $p+q = 2n$. In the event $p+q$ is odd, we will be unable to choose the two sets of roots of unity to be completely incompatible. The result will be that part of the matrix elements will be always annihilated and that will reduce the true degrees of freedom of the representation to match the Clifford algebra. Alternatively, we can bump up the Clifford algebra to $\mathcal{CL}(p+1,q)$ and then ignore the extra degrees of freedom in both the Clifford algebra and the matrices.

From the point of view of quantum mechanics, the ι_χ can represent Stern-Gerlach filters and Eq. (74) is equivalent to requiring that the observables corresponding to e_χ be completely incompatible with the observables corresponding to d_χ. Spin$-1/2$ in the x and z directions are completely incompatible:

$$\hat{x}\hat{z} = -\hat{z}\hat{x}, \tag{76}$$

and so in the case of the Pauli matrices the representation can be defined by $e_1 = \hat{z}$ and $d_1 = \hat{x}$. More generally, since the e_χ form a complete set, it is not possible to choose a d_χ that commutes with all of them and yet is not included in their degrees of freedom.

This presentation has drifted towards the theoretical so a practical application is in order. Let's write down all the Weyl representations of the Dirac algebra. That is, let's write down the representations that have the $\widehat{ixy}$ and $\widehat{zt}$ operators diagonal, and so $e_1 = \widehat{ixy}$ and $e_2 = \widehat{zt}$. How much freedom remains in the choice of the democratic primitive idempotent? The degrees of freedom that are covered by e_χ consists of the Abelian group with four elements given by the possible products of e_χ:

$$\hat{1}, \qquad \widehat{ixy}, \qquad \widehat{zt}, \qquad \widehat{ixyzt}. \tag{77}$$

The remaining degrees of freedom (written as square roots of $+1$ rather than ± 1) are:

$$\begin{array}{cccc} \hat{1} & \hat{x} & \hat{y} & \hat{z} \\ \widehat{it} & \widehat{ixz} & \widehat{xt} & \widehat{iyz} \\ \widehat{yt} & \widehat{ixyz} & \widehat{xyt} & \widehat{yzt} \end{array} \tag{78}$$

To choose the democratic primitive idempotent, we need to find the Abelian subgroups of the above. By inspection:

$$\begin{array}{l} \{\hat{1}, \hat{x}, \widehat{iyz}, +\widehat{ixyz}\}, \\ \{\hat{1}, \hat{x}, \widehat{yt}, +\widehat{xyt}\}, \\ \{\hat{1}, \hat{y}, \widehat{ixz}, -\widehat{ixyz}\}, \\ \{\hat{1}, \hat{y}, \widehat{xt}, -\widehat{xyt}\}, \\ \{\hat{1}, \hat{z}, \widehat{xt}, -\widehat{xzt}\}, \\ \{\hat{1}, \hat{z}, \widehat{yt}, -\widehat{yzt}\}, \\ \{\hat{1}, \widehat{it}, \widehat{ixz}, -\widehat{xzt}\}, \\ \{\hat{1}, \widehat{it}, \widehat{iyz}, -\widehat{yzt}\} \end{array} \tag{79}$$

For each of the eight distinct maximal Abelian subgroups, there are 6 choices for d_1. For example, in the first case, one could use $\pm\hat{x}$, $\pm\widehat{iyz}$ or $\pm\widehat{ixyz}$. Having chosen d_1, there are then 4 choices for d_2 giving a total of 24 selections for that subgroup. But the ordering of the choices makes them double counted. Thus the total number of choices for the democratic primitive idempotent is $12 \times 8 = 96$. The Weyl representation used in this paper uses elements chosen from the last of the above Abelian subgroups.

We now complete our demonstration that, with the usual choices of representations, the geometric squared magnitude $|U|^2_G$ gives identical probability predictions as the matrix squared magnitude $|U|^2_{N\times N}$, which is well known (and shown above) to have probabilities identical to the usual spinor inner product. To do this, we will follow the hint of Eq. (49) and examine the degrees of freedom corresponding to a single term in ι_D.

The way we've chosen our d_χ implies that the various terms in d_χ correspond to independent degrees of freedom even when multiplied by ι_k. In computing the geometric squared magnitude, the terms of ι_D, when multiplied by ι_j, will give stay in independent subspaces of the Clifford algebra, so these terms break the Clifford algebra into independent subspaces. This is not a general feature of representations of a Clifford algebra, but it is true, for example, of the 96 Weyl representations that we just derived and it is true of all representations that are

in common use in physics. In the example of the Weyl representation used here, the 4 subspaces generated by the terms of ι_D are spanned by the four rows in Eq. (46).

Let $\hat{\lambda}$ be a term in ι_D. Following the "coincidence" of Eq. (49) and Eq. (50), we first show that the subspace $\iota_j\ \hat{\lambda}$ gives independent degrees of freedom of the matrix (i.e. distinct matrix elements) just as it gives a geometric subspace of the Clifford algebra. Accordingly, consider the matrix elements that arise from $\hat{\lambda}$. These are just the values of j and k that leave

$$\iota_j\ \hat{\lambda}\ \iota_k \tag{80}$$

nonzero. To write ι_j in terms of e_χ, define j_m to be $+1$ or -1 according as the binary number $j-1$ as a 1 or 0 bit in its mth position. This gives:

$$\iota_j = (1+j_1e_1)(1+j_2e_2)...(1+j_ne_n)/N, \tag{81}$$

with ι_k similar and so the matrix elements that arise from $\hat{\lambda}$ are the values of j and k that leave

$$(1+j_1e_1)(1+j_2e_2)...(1+j_ne_n)/N\ \hat{\lambda}\ \iota_k. \tag{82}$$

nonzero.

We can commute $\hat{\lambda}$ around ι_j if we complement the signs of the e_χ that anticommute with $\hat{\lambda}$. Since the e_χ set of roots of unity is complete, $\hat{\lambda}$ must anticommute with at least one of the e_χ. Let κ_j be ±1 depending on whether $\hat{\lambda}$ commutes $(+1)$ or anticommutes (-1) with e_j. Then

$$\begin{array}{l}(1+j_1e_1)(1+j_2e_2)...(1+j_ne_n)/N\ \hat{\lambda} \\ =\ \hat{\lambda}\ (1+\kappa_1j_1e_1)(1+\kappa_2j_2e_2)...(1+\kappa_nj_ne_n)/N\end{array} \tag{83}$$

Substituting this into Eq. (80) we see that the matrix elements that arise from $\hat{\lambda}$ are the values of j and k that leave

$$\hat{\lambda}\ (1+\kappa_1j_1e_1)(1+\kappa_2j_2e_2)...(1+\kappa_nj_ne_n)/N\ \iota_k. \tag{84}$$

nonzero. Since the ι_j are self annihilating primitive idempotents, this will be zero unless $\kappa_1j_1 = k_1$, $\kappa_2j_2 = k_2$, ... $\kappa_nj_n = k_n$. Thus both the matrix squared magnitude and the geometric squared magnitude are compatible with the subspaces defined by ι_j multiples of the terms of ι_D. The N matrix elements that correspond to $\hat{\lambda}$ are M_{jk} where j and k are related by the restriction that $\kappa_mj_m = k_m$ for $m = 1$ to n. This defines a set of N equations in N unknowns that give the non zero matrix elements in terms of the coefficients of the canonical basis element expansion. As in Eq. (48), these N equations are easily solved to give the coefficients in terms of the matrix elements. The resulting N equations are of the form:

$$\alpha_{\chi(m)} = \frac{1}{N}\sum_{j=1}^{N} m_jA_{jk}, \tag{85}$$

where m_j are -1 or $+1$ according as the number m has a binary expansion with 0 or 1 in its jth bit position. Taking squared magnitudes on both sides and summing over m, again the cross terms on the right cancel giving the desired generalization of Eq. (49):

$$\textstyle\sum_{m=1}^{N}|\alpha_{\chi(m)}|^2\ =\ \frac{1}{N}\sum_{j=1}^{N}|A_{jk}|^2, \tag{86}$$

from which we have that the geometric squared magnitude and the matrix squared magnitude are related by $N|\alpha(A)|_G^2 = |A|_{N\times N}^2$.

VIII. ALGEBRA OF PAULI FILTERS

In this section, we begin our exploration of the algebra of projection operators with the simplest case, projection operators chosen from the Pauli algebra.

Following the formalism of the SMA, we imagine sending the output of one filter into the input of another. Earlier in this paper, we needed a formula for the phase associated with a product of three projection operators. As with the Schwinger measurement algebra, this will be modeled by multiplication, and we will make the calculation here. The effect of a Stern-Gerlach filter of the type considered here is a projection operator, so our object of study will be the overall projection defined by the product of three projection operators:

$$P = P_3\,P_2\,P_1, \tag{87}$$

where P_1 is applied first. We will use the usual Pauli notation. In this notation, the spin operator in the $\vec{u}$ direction is:

$$\sigma_u = \vec{u}\cdot\vec{\sigma} = v_x\sigma_x + v_y\sigma_y + v_z\sigma_z. \tag{88}$$

The projection operator in the u direction is given by

$$P_u = \frac{1}{2}(1+\sigma_u) = \frac{1}{2}(1+\vec{u}\cdot\vec{\sigma}). \tag{89}$$

The fundamental multiplication rule for the spin operators of the Pauli algebra is:

$$\sigma_u\sigma_v = \vec{u}\cdot\vec{v} + i(\vec{u}\times\vec{v})\cdot\vec{\sigma}. \tag{90}$$

The i of the Pauli representation can be more generally replaced with the geometric pseudoscalar:

$$\hat{i} = \sigma_x\ \sigma_y\ \sigma_z. \tag{91}$$

In a more general algebra, this $\hat{i}$ may not commute with all the elements of the algebra.

Our analysis will apply to sequences of any number of Stern-Gerlach filters, but the full results will be clear from considering just three. Accordingly, let $\vec{w}$, $\vec{v}$ and $\vec{u}$ be three unit vectors giving the spin direction of, respectively, the first, second and third Stern-Gerlach filters. The three projection operators are:

$$\begin{array}{l}P_{\chi u} = \frac{1}{2}(1+\sigma_u), \\ P_{\chi v} = \frac{1}{2}(1+\sigma_v), \\ P_{\chi w} = \frac{1}{2}(1+\sigma_w).\end{array} \tag{92}$$

With the full operator for the sequence of filters given by the product. Since this product is not in general a projection operator, we will refer to it as a filter, F_{SG}:

$$F_{SG} = P_u\, P_v\, P_w. \tag{93}$$

In order to avoid a later division by zero, we will assume that $\vec{u}$ and $\vec{w}$ are not antiparallel. That is, we will assume that $\vec{u}\cdot\vec{w} > -1$. Under this assumption, it can be shown that there is a complex constant g_{uvw} that depends on $\vec{u}$, $\vec{v}$ and $\vec{w}$ and that satisfies:

$$P_u P_v P_w = g_{uvw} P_u P_w \tag{94}$$

The existence of this constant is equivalent to noting that Stern-Gerlach filters obliterate all knowledge of previous measurements. Since the P_v measurement is hidden from the exterior of the sequence of filters by P_u and P_w, the only effect it can contribute is a change in the amplitude and phase.

The above reduction rule Eq. (94) allows us to reduce almost any product of spin projection operators to a complex constant multiplied by the leading and trailing operators. For example:

$$\begin{aligned} P_a P_b P_c P_d P_e &= g_{abc}(P_a P_c) P_d P_e \\ &= g_{abc} g_{acd} g_{ade} P_a P_e \end{aligned} \tag{95}$$

We may still be able to evaluate cases with antiparallel operators if we can arrange to eliminate the middle projection operators in an order that avoids having two antiparallel operators adjacent.

A convenient way to evaluate g_{uvw} is with traces, but we will instead use "blade" notation. Blades are vector subspaces of a real Clifford algebra. The scalars of a real Clifford algebra form the "0−blade", which are written $<>_0$. The vectors form the "1−blade". Bivectors form the "2−blade", etc. Given a real Clifford algebraic constant, the $n-$blade (i.e. $<>_n$) portion can be extracted from it. If κ is a Clifford algebraic constant, it can be written as a sum over its blade components so:

$$\kappa = \sum_n < \kappa >_n \tag{96}$$

Let M be an arbitrary element of the real Pauli algebra:

$$\begin{aligned} M \;=\; & a_0 + a_x\sigma_x + a_y\sigma_y + a_z\sigma_z \\ & + a_{ix}\sigma_y\sigma_z + a_{iy}\sigma_z\sigma_x + a_{iz}\sigma_x\sigma_y + a_i\sigma_x\sigma_y\sigma_z, \end{aligned} \tag{97}$$

where a_χ are real constants. Then the four blades are:

$$\begin{aligned} < M >_0 &= a_0 \\ < M >_1 &= a_x\sigma_x + a_y\sigma_y + a_z\sigma_z \\ < M >_2 &= a_{ix}\sigma_y\sigma_z + a_{iy}\sigma_z\sigma_x + a_{iz}\sigma_x\sigma_y \\ < M >_3 &= a_i\sigma_x\sigma_y\sigma_z. \end{aligned} \tag{98}$$

Written in the canonical basis of the Clifford algebra $\mathcal{CL}(3,0)$, the four blades are:

$$\begin{aligned} < M >_0 &= a_0 \\ < M >_1 &= a_x\hat{x} + a_y\hat{y} + a_z\hat{z} \\ < M >_2 &= a_{ix}\widehat{yz} - a_{iy}\widehat{xz} + a_{iz}\widehat{xy} \\ < M >_3 &= a_i\widehat{xyz}. \end{aligned} \tag{99}$$

Since g_{uvw} is a complex valued function, it will be convenient to use the complex Pauli algebra. There are then two blades, which we will denote by $[\ ]_n$ so that we can switch between the real and complex representations of $\mathcal{SU}(2)$:

$$[M]_0 =< M >_0 + i < M >_3 = a_0 + ia_i$$

$$[M]_1 =< M >_1 + i < M >_2 . \tag{100}$$

When the Pauli algebra is represented in 2×2 complex matrices, $[M]_0 = \mathrm{tr}(M)/2$. For example,

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]_0 = \frac{a+d}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{101}$$

We can find γ_{uvw} by examining the complex scalar portions of Eq. (94):

$$[P_u P_v P_w]_0 = g_{uvw}[P_u P_w]_0. \tag{102}$$

Using Eq. (90) and dividing by $[P_u P_w]_0$ gives:

$$g_{uvw} = \frac{1}{2} + \frac{\vec{u}+\vec{w}+i\vec{w}\times\vec{u}}{2(1+\vec{u}\cdot\vec{w})}\cdot\vec{v}. \tag{103}$$

Thus g_{uvw}, as a function of $\vec{v}$ consists of a constant $1/2$, and the dot product of $\vec{v}$ with a complex vector that depends on $\vec{u}$ and $\vec{w}$.[6]

IX. INTERFERENCE BETWEEN PRODUCTS OF PAULI FILTERS

The presence of the constant i in Eq. (103) is interesting in that it shows that we can arrange for interference using only Stern-Gerlach filters. When we pass from the wave function (spinor) formalism to the density matrix formalism, we eliminate the $\mathcal{U}(1)$ gauge freedom. Now we see that this interference is not a simple wave effect that depends on path as in the two slit experiment. Thus this interference is a physical effect associated with the Clifford algebra in which the spin $\mathcal{SU}(2)$ is embedded.

In order to better understand the interference effect that can be ascribed to Stern-Gerlach filters, let us consider a spin$-1/2$ fermion particle beam split by a beam splitter into two beams that pass through similar Stern-Gerlach composite filters. The two filters have identical first and last spin projection operators, but the central projection operators are different. After the beams pass through the Stern-Gerlach filters, they are recombined as shown in Fig. (1). The middle filter effects the interference pattern as we now calculate.

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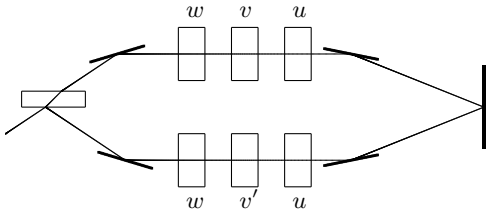
[6] The possible division by zero in Eq. (103) could be avoided by using the more general Schwinger measurement algebra symbols.

FIG. 1: A beam of spin$-1/2$ fermions begins at the left and is split into two beams by a beam splitter. The two beams are each passed through a series of 3 Stern-Gerlach filters. The first and last filter of each series are identical, but the central filter differs. After the series of filters, the two beams are brought together to generate an interference pattern. The orientation of the center filters effects the intensity of the beam and also produces interference effects detectable only in the interference pattern.



Ignoring the operation of the beam splitter, the operator F_{IE} for the interference experiment is given by the sum of two products of spin operators:

$$F_{IE} = P_u P_v P_w + P_u P_{v'} P_w. \tag{104}$$

Applying Eq. (103) we can see the interference term:

$$\begin{aligned} F_{IE} &= (\tfrac{1}{2} + \tfrac{\vec{u}+\vec{w}+i\vec{w}\times\vec{u}}{2(1+\vec{u}\cdot\vec{w})} \cdot \vec{v}) P_u P_w \\ &\quad + (\tfrac{1}{2} + \tfrac{\vec{u}+\vec{w}+i\vec{w}\times\vec{u}}{2(1+\vec{u}\cdot\vec{w})} \cdot \vec{v}') P_u P_w \\ &= (1 + \tfrac{\vec{u}+\vec{w}+i\vec{w}\times\vec{u}}{2(1+\vec{u}\cdot\vec{w})} \cdot (\vec{v}+\vec{v}')) P_u P_w \end{aligned} \tag{105}$$

If $\vec{v} = \vec{v}'$ then, while there will still be a complex rotation, there will be no interference between the two beams. The two different directions $\vec{v}$ and $\vec{v}'$ create two different paths from $\vec{u}$ to $\vec{w}$. The interference between these two different paths turns out to be proportional to the surface area of the quadrilateral on the unit sphere defined by the four points $\vec{u}, \vec{v}, \vec{w}, \vec{v}'$. To show this, it is useful to consider closed paths, that is, to consider sequences of Stern-Gerlach filters that begin and end with the same orientation.

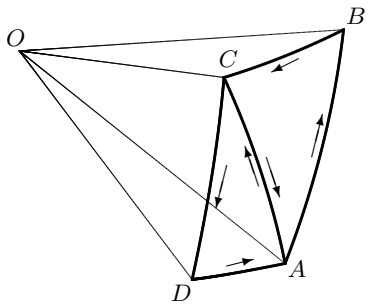
If $\vec{A} = \vec{B}$, Eq. (103) reduces to:

$$g_{ABA} = \frac{1 + \vec{A}\cdot\vec{B}}{2}. \tag{106}$$

This shows that when we consider only a single Stern-Gerlach filter placed between two identical Stern-Gerlach filters, the result is equivalent to a single lossy Stern-Gerlach filter. While these are closed paths, they enclose no surface area, so the change in phase is zero.

Let's consider the closed spin projection paths around two adjoining spherical triangles on the surface of the unit sphere: $\triangle ABC$, $\triangle ACD$ and $\diamondsuit ABCD$ as shown in Fig. (2). There are three closed paths that all start and end at A (and have the same orientation). As an operator, each is associated with a complex multiple of P_A:

FIG. 2: Two adjoining spherical triangles on the unit sphere. The complex phase parts of g_{uvw} are additive, that is, $\arg(g_{ABCD}) = \arg(g_{ABC}) + \arg(g_{ABD})$, and therefore the complex phase of a series of projection operators can be determined by the area of the spherical triangle they define.



$$\begin{aligned} F_{ABC} &= P_A P_B P_C P_A &&= g_{ABC} P_A \\ F_{ACD} &= P_A P_C P_D P_A &&= g_{ACD} P_A \\ F_{ABCD} &= P_A P_B P_C P_D P_A &&= g_{ABC} P_A P_C P_D P_A \\ & &&= g_{ABC} g_{ACD} P_A \end{aligned} \tag{107}$$

The complex multiple of P_A associated with $\diamondsuit ABCD$ is the product of the complex multiple of P_A associated with $\triangle ABC$ and the complex multiple of P_A associated with $\triangle ACD$. Thus the phases are related:

$$\arg(g_{ABC} g_{ACD}) = \arg(g_{ABC}) + \arg(g_{ACD}). \tag{108}$$

This calculation can be extended to arbitrary shapes, so the phase of a closed oriented path of spin projection operators must be proportional to the surface area bounded by the path. The constant of proportionality can be found by looking at infinitesimal paths and is $1/2$. The resulting formula for the phase of a product of spin projection operators is:

$$\begin{aligned} &\arg(g_{ABC} g_{ACD} g_{ADE} \cdots) \\ &= \arg(g_{ABC}) + \arg(g_{ACD}) + \arg(g_{ACE}) + \ldots \\ &= \tfrac{1}{2}(\text{Spherical area defined by path } ABCDE\ldots), \end{aligned} \tag{109}$$

Thus there is a geometric relationship between phase changes of spin projection operators and paths through which they are rotated. In Section sec:PaPiPF we will elaborate this relationship.

If we consider a path defined by spin operators that goes around the equator of the unit sphere, the surface area cut by the path will be $4\pi/2 = 2\pi$, and the complex phase associated will be $2\pi/2 = \pi$. For example, let the sequence of orientation vectors be $\hat{z}$, $\hat{x}$, $-\hat{z}$, $-\hat{x}$ and $\hat{z}$.

The operator for this sequence of Stern-Gerlach filters is:

$$\begin{aligned} F &= P_z P_x P_{-z} P_{-x} P_z \\ &= \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= -\frac{1}{4}\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = -\frac{1}{4}P_z \end{aligned} \tag{110}$$

The overall effect of the above filter is to multiply the $S_z = +1/2$ component of a spinor by $-1/4$ and to annihilate the $S_z = -1/2$ component. Thus the amplitude will decrease by a factor of 16 and a phase rotation of π will be effected.

It is a well known fact of quantum mechanics that when a spinor is rotated through an angle of 2π using another spinor, the rotated spinor obtains a phase angle of π or is multiplied by -1. From the viewpoint of the density matrix formalism, this is a consequence not of some obscure feature of the electron, but instead is a consequence of the gauge freedom introduced in factoring the natural Banach space into an unphysical Hilbert space.

X. PHASES AND PROBABILITIES OF PROJECTION OPERATORS

In order to further explore the relationship between phases and probabilities in products of projection operators, we now introduce three unit vectors, $\vec{r}$, $\vec{g}$, $\vec{b}$. Phases are associated with products of projection operators that are not coplanar. For simplicity, we will assume that the three vectors make up an equilateral spherical triangle. That is, we assume that

$$\vec{r}\cdot\vec{g} = \vec{g}\cdot\vec{b} = \vec{b}\cdot\vec{r} = \cos(\theta_\angle), \tag{111}$$

where $\theta_\angle$ is the opening angle between any pair of the vectors. We now further explore the relationship between probabilities and the complex phase (and generalizations of the complex phase) for projection operators that form equilateral triangles in 3−space.

Our eventual objective is to analyze products of projection operators of the form $\hat{R}\hat{G}\hat{B}$. We will begin by analyzing the Pauli algebra situation. This will use the standard probability structure of quantum mechanics. Accordingly, define the projection operators:

$$\begin{aligned} P_r &= (1+\sigma_r)/2 = (1+\vec{r}\cdot\vec{\sigma}), \\ P_g &= (1+\sigma_g)/2, \\ P_b &= (1+\sigma_b)/2, \end{aligned} \tag{112}$$

we desire to calculate the product:

$$F_{rgbr} = P_r\ P_g\ P_b\ P_r = T(rgbr)\ e^{i\phi(rgb)}\ P_r, \tag{113}$$

where $T(rgb)$ is the (real) amplitude associated with the sequence and $\phi(rgb)$ is the phase. We expect that $T(rgb)$ will factor into two parts, each associated with the losses in traversing from one filter to another:

$$T_(rgbr) = T_{RG}\ T_{GB}\ T_{BR}. \tag{114}$$

For the usual Pauli algebra, T_{RG} is simply

$$T_{RG} = T_{GB} = T_{RB} = \sqrt{(1+\cos(\theta_\alpha))/2}. \tag{115}$$

In later sections this paper will generalize the Pauli algebra so as to form probabilities larger than this, but for now, let us first carefully examine the probabilities of the Pauli algebra.

The overall probability, $P(rgb) = T^2(rgbr)$ is given by the product of three identical independent transitions, each with probability $(1+\cos(\theta_\angle))/2$. The overall probability is therefore:

$$P(rgbr) = \frac{1}{8}(1+\cos(\theta_\angle))^3, \tag{116}$$

and the amplitude is just the square root of this:

$$T(rgbr) = (0.5 + 0.5\cos(\theta_\angle))^{1.5}. \tag{117}$$

The phase, according to Eq. (109), is half the area of the spherical triangle defined by $\vec{r}$, $\vec{g}$, and $\vec{b}$. The area of a spherical triangle is the excess over π of the sum of its corner angles. Thus

$$\phi(rgb) = (3\theta_c - \pi)/2, \tag{118}$$

where θ_c is a corner angle. Applying the law of cosines for spherical triangles gives the following relationship between θ_c, the angles of the spherical equilateral triangle, and $\theta_\angle$ the sides of the triangle:

$$\cos(\theta_\angle) = \cos^2(\theta_\angle) + \sin^2(\theta_\angle)\cos(\theta_c). \tag{119}$$

substituting this into Eq. (118) and converting sines into cosines gives:

$$\phi(rgb) = \tfrac{3}{2}\cos^{-1}(\cos(\theta_\angle)/(1+\cos(\theta_\angle))) - \tfrac{\pi}{2}. \tag{120}$$

Combining this with Eq. (117) gives the relationship between phase and probability for equilateral spherical triangles as:

$$\begin{aligned} \phi &= \tfrac{3}{2}\cos^{-1}\left(1 - \tfrac{1}{2(T(rgbr))^{2/3}}\right) - \tfrac{\pi}{2}, \\ &= \tfrac{3}{2}\cos^{-1}\left(1 - \tfrac{1}{2P^{1/3}}\right) - \tfrac{\pi}{2}. \end{aligned} \tag{121}$$

There are several important special cases. When $P = 1$ one finds that $\phi = 0$ as is appropriate for three vectors in the same direction. When $P = (0.5)^3 = 1/8$, the three vectors are perpendicular and $\phi = \pi/4$. When $P = (0.25)^3 = 1/64$, the three vectors are coplanar, 120 degrees apart, and $\phi = \pi$.

XI. EXPONENTIAL MAPS AND PARTICLE MODELS

A convenient method of obtaining nonstandard representations of Clifford algebras, and therefore representations where the geometric squared magnitude may give

a different probability from the usual, is to use the exponential of a complex multiple of a canonical basis element. To demonstrate this, consider the following mapping of the Clifford algebra $\mathcal{CL}(3,1)$:

$$\begin{aligned}
\hat{x} \to \hat{x}' &= e^{-\alpha\widehat{zt}/2}\ \hat{x}\ e^{+\alpha\widehat{zt}/2}, \\
&= \hat{x}, \\
\hat{y} \to \hat{y}' &= e^{-\alpha\widehat{zt}/2}\ \hat{y}\ e^{+\alpha\widehat{zt}/2}, \\
&= \hat{y}, \\
\hat{z} \to \hat{z}' &= e^{-\alpha\widehat{zt}/2}\ \hat{z}\ e^{+\alpha\widehat{zt}/2}, \\
&= \hat{z}e^{+\alpha\widehat{zt}}, \\
\hat{t} \to \hat{t}' &= e^{-\alpha\widehat{zt}/2}\ \hat{t}\ e^{+\alpha\widehat{zt}/2}, \\
&= \hat{t}e^{+\alpha\widehat{zt}},
\end{aligned} \tag{122}$$

where α is a real (or complex) constant. The mapping clearly preserves multiplication and addition, so the set $\{\hat{x}', \hat{y}', \hat{z}', \hat{t}'\}$ also forms a basis for the Clifford algebra $\mathcal{CL}(3,1)$. Thus we can modify a given matrix representation of a Clifford algebra by using the exponential map. That the new representation carries a different probability interpretation is seen by noting that

$$|\hat{z}'|_G^2 = |\hat{t}'|_G^2 = \cosh^2(\alpha) + \sinh^2(\alpha) = \cosh(2\alpha), \tag{123}$$

while $|\hat{z}|_G^2 = |\hat{t}|_G^2 = 1$, and that the alteration also occurs in the primitive idempotents $(1 \pm \hat{z})/2$ and $(1 \pm \hat{t})/2$. If, in the above example, we had instead chose $\exp(\alpha\widehat{xy})$, while we would have found that $\hat{x}$ and $\hat{y}$ were remapped, we would still have $|\hat{x}'|_G^2 = |\hat{x}|_G^2$, etc., and thus the probability interpretation would be unchanged.

If we are to restrict our attention to representations of Clifford algebras that treat the spatial dimensions, $\hat{x}$, $\hat{y}$, and $\hat{z}$, equivalently, then we can only exponentiate with canonical basis elements that have no spatial orientation. Among the Dirac algebra (i.e. the complex $\mathcal{CL}(3,1)$), this restricts our attention to exponentiation by $\widehat{xyz}, \hat{t}$, and $\widehat{xyzt}$, or the imaginary multiples of these. Half of these six possibilities square to -1 and therefore correspond to rotations that leave probabilities unchanged:

$$\begin{array}{cccc}
 & \widehat{xyz} & \hat{t} & \widehat{xyzt} \\
\hat{x} \to & \hat{x} & c\hat{x} + s\widehat{xt} & c\hat{x} + s\widehat{yzt} \\
\hat{y} \to & \hat{y} & c\hat{y} + s\widehat{yt} & c\hat{y} - s\widehat{xzt} \\
\hat{z} \to & \hat{z} & c\hat{z} + s\widehat{zt} & c\hat{z} + s\widehat{xyt} \\
\hat{t} \to & c\hat{t} - s\widehat{xyzt} & \hat{t} & c\hat{t} + s\widehat{xyz}
\end{array} \tag{124}$$

where c and s stand for the cosine and sine. The rotations that modify the $|\ \ |_G^2$ are almost the same:

$$\begin{array}{cccc}
 & \widehat{ixyz} & \widehat{it} & \widehat{ixyzt} \\
\hat{x} \to & \hat{x} & c\hat{x} + s\widehat{ixt} & c\hat{x} + s\widehat{iyzt} \\
\hat{y} \to & \hat{y} & c\hat{y} + s\widehat{iyt} & c\hat{y} - s\widehat{ixzt} \\
\hat{z} \to & \hat{z} & c\hat{z} + s\widehat{izt} & c\hat{z} + s\widehat{ixyt} \\
\hat{t} \to & c\hat{t} - s\widehat{ixyzt} & \hat{t} & c\hat{t} + s\widehat{ixyz}
\end{array} \tag{125}$$

but with c and s standing for the hyperbolic trigonometric functions. One can also repeatedly apply different exponential maps. It should be noted that if a canonical basis element happens to commute with all the rest of the algebra, the exponential map will be the identity. We will ignore these trivial maps.

If two canonical basis vectors are modified by an exponential map, one finds that the product of the two basis vectors is unmodified. For example:

$$\begin{aligned}
\widehat{xt}' &= (\hat{x}')(\hat{t}'), \\
&= (\hat{x}e^{+\alpha\widehat{ixyzt}})(\hat{t}e^{+\alpha\widehat{ixyzt}}), \\
&= \hat{x}\hat{t}e^{-\alpha\widehat{ixyzt}}e^{+\alpha\widehat{ixyzt}}), \\
&= \widehat{xt}.
\end{aligned} \tag{126}$$

The product of a modified vector by an unmodified vector is a bivector that is modified, while the product of two unmodified objects is also unmodified. Thus the property of "is modified by the exponential map χ" divides the Clifford algebra into two subspaces, one of which is modified while the other is unmodified. Of course 1 always shows up in the unmodified half.

As an illustration of the exponential modification of a representation, and how this produces a modification of the probability postulate, we apply the $\alpha\widehat{ixyzt}$ exponential map to the Weyl representation. This exponential map has the attribute that it does not modify any of the diagonalized operators. Of the four rows of matrices in Eq. (46), the middle two are modified while the top and bottom rows are not. Thus the 16 complex degrees of freedom of the Clifford algebra are split into two groups of 8 according to a 2×2 checkerboard pattern:

$$\begin{pmatrix} 1 & 1 & c & c \\ 1 & 1 & c & c \\ c & c & 1 & 1 \\ c & c & 1 & 1 \end{pmatrix}, \tag{127}$$

where 1 designates the entries that are unmodified, while c designates the entries that are modified. The effect of the modification is to increase the relative intensities of the modified entries.

In the Weyl representation of the Dirac algebra, the diagonal primitive idempotents are operators that pick out the four components of a spinor. The four components correspond to the left and right handed electrons and positrons and the spinor is composed as follows:[12, §3.3]

$$\begin{pmatrix} \bar{e}_L \\ e_L \\ e_R \\ \bar{e}_R \end{pmatrix} \tag{128}$$

Earlier in this paper, we noted that square spinors provide a method of combining multiple particles into a single Dirac equation. In that context, as a toy model, we could interpret the 4 columns in the above matrix as four different spin$-1/2$ particles.[7] Naming these particles a,

[7] This follows Trayling, [11] [4] who puts the eight elementary

b, c, and d, we could suppose that Eq. (127) implies that these four particles have components that can be characterized as having a_R, $\bar{a}_R$, b_R, $\bar{b}_R$, $\bar{c}_L$, c_L, $\bar{d}_L$, and d_L more intense (i.e. associated with the modified elements of the matrix), and the remaining 8 particles less intense. There are many possible ways that one might arrange the particles, and with the addition of an extra dimension (or three or four, as in the case with Trayling[11]), we could possibly obtain a model of the elementary fermions. However, this would be inconsistent with our assumption of the use of the density matrix formalism and the Schwinger measurement algebra.

As an alternative toy model, we could suppose that the 4×4 Dirac matrix can provide us with only one particle, that particle being composed of left and right handed components, and particle and antiparticle components in the usual manner. That is, the various components of the electron, e_L, e_R, $\bar{e}_L$, and $\bar{e}_R$ are to be associated with the diagonal primitive idempotents. The off diagonal elements then are associated with interactions between these components.

For the choice of $\alpha\widehat{ixyzt}$, the modified interactions correspond to interactions that relate a left-handed particle to a right-handed one, while the unmodified interactions relate to interactions that preserve handedness. Again, the addition of extra dimensions provides some leniency that may allow modeling of the elementary particles, but the author believes that a more profitable method is to assume a preon model that will be described in a later paper.

A conceptual difficulty in associating the elementary particles with the primitive idempotents (in particular, the diagonal matrix elements) of a Clifford algebra, and the elementary interactions with the nilpotents (off diagonal matrix elements), is that because of the Pauli exclusion principle we expect the fermions to satisfy an anticommuting Grassmann algebra with $\psi^\dagger \psi^\dagger = 0$, while the particle interactions are modeled as bosons where the creation operators commute. Thus the algebraic assumptions in our second toy model are apparently in reverse of the usual.

But our usual algebraic assumption of nilpotency for these particles appears in the spinor representation, where our intuition has been well honed by long practice, and in this paper we are explicitly working in the density matrix formalism. Thus after converting from spinor form to density matrix form, the creation and annihilation operators for a state are idempotent:

$$\begin{aligned} \rho &= \psi\, \psi^\dagger, \\ \rho\, \rho &= \psi\, \psi^\dagger\, \psi\, \psi^\dagger, \\ &= \psi\, \psi^\dagger. \end{aligned} \tag{129}$$

fermions of the electron family, e, ν_e, d_R, d_G, d_B, u_R, u_G, and u_B, along with handedness and particle/antiparticle into the Clifford algebra $\mathcal{CL}(7,0)$.

In treating the bosons as interactions between fermions, the Pauli principle shows that it is impossible to simultaneously create two bosons using identical fermions, however, there is no restriction on creating two apparently identical bosons by two distinct fermion interactions. Thus the bosons satisfy Bose statistics due to the requirement that the fermions that create and annihilate them be separated. The fact that all the bosons are shared by the three families of fermions suggests that the three families are related, a topic we will continue in the next section.

XII. HIDDEN DIMENSIONS AND PROBABILITIES.

This section introduces physically relevant and interesting, examples of modifications of the probability rule that were described earlier. We will be dealing with phases and probabilities in modifications of the Pauli algebra that can be attributed to the presence of hidden dimensions. Let $\hat{s}$ and $\hat{t}$ be considered as a hidden spatial and temporal dimension, with signatures $+1$ and -1, respectively. We will be considering these as extensions of the Pauli algebra. Note that by setting $\hat{s} = i\widehat{xyzt}$, we can arrange for these examples to be exhibited by the Dirac algebra.

The probability of transition between two spinors that are oriented in different directions $\vec{u}$ and $\vec{v}$ with an angle of θ between them is:

$$\begin{aligned} P_\theta &= \mathrm{tr}\left(\tfrac{1+\vec{u}\cdot\vec{\sigma}}{2}\tfrac{1+\vec{v}\cdot\vec{\sigma}}{2}\right), \\ &= \tfrac{1}{4}\mathrm{tr}(1+\hat{u}+\hat{v}+\hat{u}\hat{v}), \\ &= \tfrac{1}{4}\mathrm{tr}(1+\vec{u}\cdot\vec{v}), \\ &= \tfrac{1}{4}\, 2(1+\cos(\theta)) = \tfrac{1+\cos(\theta)}{2}, \end{aligned} \tag{130}$$

where we have changed notation from the Pauli algebra to the Clifford algebra halfway through the calculation as a reminder of the notation to the reader, used the fact that $\mathrm{tr}(1) = 2$, and that

$$\hat{u}\hat{v} = \vec{u}\cdot\vec{v} + (\vec{u}\times\vec{v})\cdot\vec{\sigma}. \tag{131}$$

This calculation has been made the traditional way, using the trace function. We now redo the calculation with more attention paid to the details, but using the geometric definition of the trace in the density matrix formalism.

We begin with two operators, $\hat{u}$ and $\hat{v}$. These operators are roots of unity:

$$\hat{u}\,\hat{u} = 1, \qquad \hat{v}\,\hat{v} = 1, \tag{132}$$

and therefore they possess eigenvectors with eigenvalues of ± 1. We are concerned with the eigenvectors that possess eigenvalues of $+1$. If we were to solve this equation with spinors, we would find these eigenvectors by solving the eigenvector equation, and then convert them to density matrix form by multiplying by their Hermitian conjugate. The result would be an element of the Pauli

algebra. It is easier, however, to note that the eigenvectors with eigenvalues +1 are obvious and can be written down much more easily by simply noting:

$$\begin{array}{rcl} \hat{u}\,\frac{1+\hat{u}}{2} & = & \frac{1+\hat{u}}{2}, \quad \text{or} \\ \hat{u}\,\rho_u & = & +\rho_u, \end{array} \tag{133}$$

where ρ_u is the density matrix corresponding to the spinor that has spin $+1/2$ in the $\vec{u}$ direction. With the form of these density matrices established, the transition probability can be computed as twice the 0−blade:

$$P_\theta = 2\langle (1+\hat{u})(1+\hat{v})/4 \rangle_0. \tag{134}$$

Since we have a geometric theory that is rotationally invariant, let us choose convenient coordinates and arrange for $\hat{u}$ and $\hat{v}$ to be convenient:

$$\begin{array}{rcl} \hat{u} & = & \hat{x}, \\ \hat{v} & = & \cos(\theta)\hat{x} + \sin(\theta)\hat{y}. \end{array} \tag{135}$$

Then the blade calculation for the probability is:

$$P_\theta = 2\langle (1+\hat{x})(1+\cos(\theta)\hat{x}+\sin(\theta)\hat{y})/4 \rangle_0 = (1+\cos(\theta))/2. \tag{136}$$

It should be noted that the equivalent calculation using spinors and the usual representation of the Pauli algebra is rather more involved.

If $P_\theta = 0.5$, the two directions $\vec{u}$ and $\vec{v}$ are independent. This happens when $\cos(\theta) = 0$, or

$$\theta_{0.5} = \pi/2. \tag{137}$$

If our waves were classical, we would expect that setting $\theta = \pi/2$ would leave the two waves independent. In quantum mechanics, instead this means that the two waves are independent in that the transition probabilities are 50%. For the Pauli algebra, three directions exist that are mutually independent, and these form 90 degree angles with each other.

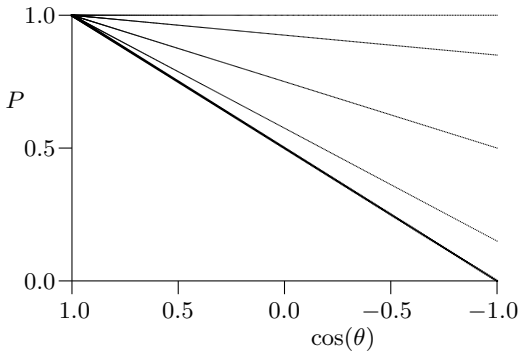
In the above paragraph, we showed that in geometric form, we can take two operators that square to 1, derive their +1 eigenvectors (or states), and compute the angle between the two states very easily in the geometric language. Nothing in the above made any assumptions about the operators $\hat{u}$ and $\hat{v}$ other than that they square to unity. As was shown in Section (VII), the geometric trace gives the same results as the usual theory in cases where the roots of unity used to define the spinor are conventional.

We now repeat the calculation with the assumption that the two operators (and therefore their states) share a little of the hidden dimension $\hat{s}$. Define

$$\hat{u}_s = \cos(\alpha)\hat{u} + \sin(\alpha)\hat{s}, \tag{138}$$

where α is a real number. This is an example of the transformations of the canonical basis elements by exponentiation. In this case, one obtains the transformation by applying the transforming functions are of the form $\exp(0.5\alpha\widehat{xs})$, as in Eq. (122), and then transformed twice more with $\hat{x}$ replaced by $\hat{y}$ and $\hat{t}$.

FIG. 3: Transition probabilities $P_{\theta s}$ between $\hat{u}_s = \cos(\alpha)\hat{u} + \sin(\alpha)\hat{s}$ and $\hat{v}_s$ for $\alpha = n\pi/8, n = 0, 1, 2, 3, 4$. The usual probability, $P_\theta = (1 + \cos(\theta))/2$ is shown as the darker line.



As with the exponential transformations, we have that $\hat{u}_s\,\hat{u}_s = 1$, and we can compute the transition probabilities between the density matrix of the $+1/2$ states of the $\hat{u}_s$ operators as:

$$\begin{array}{rcl} P_{\theta s} & = & 2\langle (1+\hat{u}_s)(1+\hat{u}_s)/4 \rangle_0, \\ & = & 2\langle (1+\cos(\alpha)\hat{x}+\sin(\alpha)\hat{s})(1+\cos(\alpha)\cos(\theta)\hat{x} \\ & & +\cos(\alpha)\sin(\theta)\hat{y}+\sin(\alpha)\hat{s})/4 \rangle_0, \\ & = & 2(1+\sin^2(\alpha)) + \cos^2(\alpha)\cos(\theta) + /4, \\ & = & \frac{1}{2}((1+\sin^2(\alpha)) + \cos(\theta)(1-\sin^2(\alpha)). \end{array} \tag{139}$$

Setting $\theta = 0$, we get that the probability is 1. Setting $\theta = \pi$, we get the minimum value of $P_{\theta s}$ as:

$$P_{\pi s} = (1 - \cos(2\alpha))/2. \tag{140}$$

When $\alpha = 0$, we get the usual result that the minimum probability is zero when the vectors are in opposite directions.

The minimum probability increases as α increases until at $\alpha = \pi/2$, it is, as expected, stuck at one. Drawn as a function of $\cos(\theta)$, we have that $P_{\theta s}$ becomes flatter as α increases. See Fig. (3) for an illustration. This is consistent with our physical intuition. For the two states to be independent, the angle between them must satisfy:

$$\begin{array}{rcl} 1/2 & = & (1+\sin^2(\alpha)+\cos^2(\alpha)\cos(\theta))/2, \quad \text{so} \\ \theta_{0.5s} & = & \cos^{-1}(-\sin^2(\alpha)/\cos^2(\alpha)), \\ & = & \cos^{-1}(-\tan^2(\alpha)). \end{array} \tag{141}$$

This is an angle greater than 90 degrees. In order to allow three independent directions to be mutually independent, we must have that $\theta_{0.5s} \leq 2\pi/3$.

Let us repeat the exercise, this time adding a little of the hidden dimension $\hat{t}$ instead of $\hat{s}$ to the two operators. As before, define

$$\hat{u}_t = \cosh(\alpha)\hat{u} + \sinh(\alpha)\hat{t}, \tag{142}$$

where α is a real number. As before, $\hat{u}_t\ \hat{u}_t = 1$ and the calculation goes through with $\hat{u}_t$ and $\hat{v}_t$.

$$\begin{aligned} P_{\theta t} &= 2\langle(1+\hat{u}_t)(1+\hat{u}_t)/4\rangle_0, \\ &= 2\langle(1+\cosh(\alpha)\hat{x}+\sinh(\alpha)\hat{t})(1+\cosh(\alpha)\cos(\theta)\hat{x} \\ &\quad +\cosh(\alpha)\sin(\theta)\hat{y}+\sinh(\alpha)\hat{s})/4\rangle_0, \\ &= 2(1+\cosh^2(\alpha)\cos(\theta)-\sinh^2(\alpha))/4, \\ &= (1-\sinh^2(\alpha)+\cosh^2(\alpha)\cos(\theta))/2. \quad \text{(bad)} \end{aligned} \tag{143}$$

When $\theta = 0$, this gives the usual result $P_{\theta t} = 1$, but when $\theta = \pi$, we get a minimum value for $P_{\theta t}$ which is negative:

$$P_{\pi t} = (1-\cosh(2\alpha))/2. \quad \text{(bad)} \tag{144}$$

Of course this is in violation of our physical intuition. Negative probabilities are bad enough, but the two states become more distinct the larger the amount that they share (i.e. as α gets larger, $P_{\theta t}$ gets smaller).

Evidently, this is an opportunity for us to use the new probability assumption introduced in this paper. Therefore let us repeat the calculation for $P_{\theta t}$, but using the $|\ \ |^2_G$, instead of the trace. The calculation will be as follows:

$$P_{\theta t} = |\rho_u\rho_v|^2_G/|\rho_u|^2_G = |\rho_u\rho_v|^2_G/|\rho_v|^2_G. \tag{145}$$

First, computing $|\rho_u|^2_G$, we find that:

$$\begin{aligned} |\rho_u|^2_G &= |(1+\cosh(\beta)\hat{u}+\sinh(\beta)\hat{t})/2|^2_G, \\ &= (1+\cosh^2(\beta)+\sinh^2(\beta))/4, \\ &= \cosh^2(\beta)/2. \end{aligned} \tag{146}$$

The formulas get long, so we abbreviate $\cosh(\beta) = \beta_c$, $\sinh(\beta) = \beta_s$, $\cos(\theta) = c$, $\sin(\theta) = s$. The computation for $|\rho_u\rho_v|^2_G$ is as follows:

$$\begin{aligned} |\rho_u\rho_v|^2_G &= |(1+\hat{u}_t)(1+\hat{v}_t)/4\|^2_G, \\ &= |(1+\beta_c\hat{x}+\beta_s\hat{t}) \\ &\quad (1+c\beta_c\hat{x}+s\beta_c\hat{y}+\beta_s\hat{t})/4|^2_G, \end{aligned}$$

$$\begin{aligned} &= \tfrac{1}{16}|(1+c\beta_c^2-\beta_s^2)+(\beta_c+c\beta_c)\hat{x}+(s\beta_c)\hat{y} \\ &\quad +(2\beta_s)\hat{t}+(\beta_c\beta_s-c\beta_c\beta_s)\widehat{xt}+(-s\beta_c\beta_s)\widehat{yt} \\ &\quad +(s\beta_c^2)\widehat{xy}|^2_G, \\ &= \tfrac{1}{16}((1+c^2\beta_c^4+\beta_s^4+2c\beta_c^2-2\beta_s^2-2c\beta_c^2\beta_s^2) \\ &\quad +(\beta_c^2+c^2\beta_c^2+2c\beta_c^2)+(s^2\beta_c^2)+(4\beta_s^2) \\ &\quad +(\beta_c^2\beta_s^2+c^2\beta_c^2\beta_s^2-2c\beta_c^2\beta_s^2)+(s^2\beta_c^2\beta_s^2) \\ &\quad +(s^2\beta_c^4)), \end{aligned}$$

$$\begin{aligned} &= \tfrac{1}{16}((1+\beta_c^4+\beta_s^4+2\beta_c^2+2\beta_s^2+2\beta_c^2\beta_s^2) \\ &\quad +c(4\beta_c^2-4\beta_c^2\beta_s^2)), \\ &= \tfrac{1}{16}((2+2\beta_c^2+2\beta_s^2+4\beta_c^2\beta_s^2) \\ &\quad +c(4\beta_c^2-4\beta_c^2\beta_s^2)), \\ &= \tfrac{1}{8}((2\beta_c^2+2\beta_c^2\beta_s^2)+c(2\beta_c^2-2\beta_c^2\beta_s^2), \\ &= \tfrac{\beta_c^2}{4}(\beta_c^2+c(1-\beta_s^2)), \end{aligned} \tag{147}$$

Finally,

$$\begin{aligned} P_{\theta t} &= |\rho_u\rho_v|^2_G/|\rho_u|^2_G, \\ &= \tfrac{1}{2}((1+\sinh^2(\beta))+\cos(\theta)(1-\sinh^2(\beta))), \end{aligned} \tag{148}$$

As expected, it is an increasing function of β and is 1 for $\theta = 0$. Putting $\beta = 0$ gives the usual Pauli probability of $(1+\cos(\theta))/2$, while setting $\theta = \pi$ gives $P_{\pi t} = \sinh^2(\beta)$. Therefore, we have a probability interpretation only so long as $|\beta| < \sinh^{-1}(1) = 0.88137$, with larger values giving probabilities greater than one. The value of θ that corresponds to independent probabilities is

$$\theta_{0.5t} = \cos^{-1}(-\sinh^2(\beta)/(1-\sinh^2(\beta))). \tag{149}$$

XIII. PHASES IN MODIFIED PAULI ALGEBRAS

As we will see in later sections, sets of three vectors that correspond to mutually independent probabilities (i.e. $\theta_{0.5s}$ or $\theta_{0.5t}$) are useful in modeling the elementary fermions. Intuitively, these correspond to mutually indpendent coordinate systems that are generalized to the presence of a hidden dimension. Phases in the Pauli matrices are handled easily because of the convenient fact that $\sigma_x\sigma_y\sigma_z = i$. However, many Clifford algebras do not possess a natural imaginary unit, that is, an element that squares to -1 and commutes with everything in the algebra.

The previous section considered modifications of the type:

$$\sigma_u \to \sigma'_u = \cos(\alpha)\hat{u}+\sin(\alpha)\hat{s}, \tag{150}$$

where α was a real parameter. More generally, let us consider modifications of the Pauli algebra of the sort:

$$\sigma_u \to \sigma'_u = A\hat{u}+B, \tag{151}$$

where $\hat{u}$ is a basis vector from the Pauli algebra (i.e. $\hat{u} = u_x\hat{x}+u_y\hat{y}+u_z\hat{z}$) and A and B are Clifford algebraic constants that have no directional content, but such that σ'_u still satisfies the usual spin$-1/2$ operator equation:

$$\sigma'_u\ \sigma'_u = (A\hat{u}+B)^2 = 1, \tag{152}$$

and therefore we can trivially solve the eigenvector equation:

$$\frac{\sigma'_u}{2}\ (1+\sigma'_u)/2 = \frac{1}{2}(1+\sigma'_u)/2; \tag{153}$$

that is, the analogue of the spin$-1/2$ eigenvector equation in the $\vec{u}$ direction.

Examples of A and B that satisfy Eq. (152) include:

$$\begin{array}{rccccc} A &= (\cos(\alpha)), & (\hat{1}+\hat{t}+\widehat{xyzt}), & (\hat{t}+\widehat{st}), & (\widehat{st}+\widehat{xyzt}), \\ B &= (\sin(\alpha)\hat{s}), & 0, & (\hat{1}+frm[o]--\hat{s}), & (\hat{1}+\widehat{xyzs}), \end{array} \tag{154}$$

where we have left the reader to supply most of the real factors necessary to arrange for normalization as discussed below. The first column gives the $\hat{s}$ example discussed in the previous section.

Since A and B are required to have no orientation, we can factor $\hat{u}$ through them, leaving them changed to $\tilde{A}$ and $\tilde{B}$:

$$A\hat{u} = \hat{u}\tilde{A}, \qquad \hat{u}B = \tilde{B}\hat{u}. \tag{155}$$

For example:

$$(\hat{1} + \hat{t} + \widehat{xyzt})\,\hat{u} = \hat{u}\,(\hat{1} - \hat{t} - \widehat{xyzt}), \tag{156}$$

so if $A = (\hat{1} + \hat{t} + \widehat{xyzt})$, then $\tilde{A} = (\hat{1} - \hat{t} - \widehat{xyzt})$. Clearly, tilde squares to unity:

$$\tilde{\tilde{A}} = A. \tag{157}$$

With this notation, Eq. (152) becomes:

$$\begin{aligned} 1 &= (A\hat{u} + B)^2, \\ &= (A\hat{u} + B)(A\hat{u} + B), \\ &= A\hat{u}A\hat{u} + BA\hat{u} + A\hat{u}B + BB), \\ &= A\tilde{A}\hat{u}\hat{u} + BA\hat{u} + A\tilde{B}\hat{u} + BB, \\ &= (A\tilde{A} + BB) + (BA + A\tilde{B})\hat{u}. \end{aligned} \tag{158}$$

Which we can achieve by arranging for:

$$A\tilde{A} + BB = 1, \qquad BA + A\tilde{B} = 0. \tag{159}$$

These relations will allow the reader to quickly derive the normalization constants needed in Eq. (154).

As in Section (IX), to get phases we will consider products of projection operators that define a triangle in the unit sphere. In order to match notation with applications to the elementary fermions, we will define the three unit vectors in color notation. We need three vectors that are equidistant from each other and that are independent in the sense that the transition probabilities between them give 50%. To make the transition probabilities equal, the three unit vectors, $\vec{r}$, $\vec{g}$ and $\vec{b}$, must be distributed equidistant on a cone.

We define the angle between two of the vectors as $\theta_\angle$:

$$\vec{r}\cdot\vec{g} = \vec{r}\cdot\vec{b} = \vec{g}\cdot\vec{b} = \cos(\theta_\angle), \tag{160}$$

so $0 < \theta_\angle \leq 2\pi/3$. The three vectors lie equidistant on a cone that has an opening angle of θ_b, where b stands for binon. The relation between the cone opening angle and the angle between two of the vectors is:

$$\cos(\theta_\angle) = \cos^2(\theta_b) - \sin^2(\theta_b)/2, \tag{161}$$

as a small amount of trigonometry will show. For compact equations, we will abbreviate:

$$\cos(\theta_\angle) = c_\angle, \qquad \sin(\theta_\angle) = s_\angle, \tag{162}$$

and similarly for c_b and s_b.

In order to simplify calculations, we choose the $z-$axis oriented in the direction of the red vector, and put the $x-$axis so that the green vector is in the $x-z$ plane with the blue vector on the $+y$ side. Explicit coordinates for the three vectors is thus:

$$\begin{aligned} \vec{r} &= (\ \ 0,\ \ 0,\ \ 1), \\ \vec{g} &= (\ s_\angle,\ \ 0,\ c_\angle), \\ \vec{b} &= (\ \ b_x\ \ b_y,\ c_\angle), \end{aligned} \tag{163}$$

where $b_x = c_\angle(1 - c_\angle)/s_\angle$ and $b_y = \sqrt{(b_x^2 + s_\angle^2)}$. The canonical basis vectors (which are also just Pauli algebra spin$-1/2$ operators in their various directions) associated with $\vec{r}$, $\vec{g}$ and $\vec{b}$ are:

$$\begin{aligned} \sigma_r &= \hat{z}, \\ \sigma_g &= s_\angle\hat{x} + c_\angle\hat{z}, \\ \sigma_b &= b_x\hat{x} + b_y\hat{y} + c_\angle\hat{z}. \end{aligned} \tag{164}$$

The modified Pauli operators are:

$$\begin{aligned} \hat{r} &= A\sigma_r + B, \\ \hat{g} &= A\sigma_g + B, \\ \hat{b} &= A\sigma_b + B, \end{aligned} \tag{165}$$

and the corresponding projection operators are:

$$\begin{aligned} \hat{R} &= (1 + A\sigma_r + B)/2, \\ \hat{G} &= (1 + A\sigma_g + B)/2, \\ \hat{B} &= (1 + A\sigma_b + B)/2, \end{aligned} \tag{166}$$

We now consider products of the form $\hat{R}\hat{G}\hat{R}$ and $\hat{R}\hat{G}\hat{B}\hat{R}$.

First, let us suppose that $B = 0$, and so, by Eq. (159), we have $A\tilde{A} = \tilde{A}A = 1$. These cases are analogous to the usual Pauli case, but without the convenience of $\sigma_x\sigma_y\sigma_z = i$ as we now show. Computing $\hat{R}\hat{G}\hat{R}$, the hard way, we find that the result is a real multiple of $\hat{R}$:

$$\begin{aligned} \hat{R}\hat{G}\hat{R} &= \tfrac{1}{8}(1 + A\hat{z})(1 + c_\angle A\hat{z} + s_\angle A\hat{x})(1 + A\hat{z}), \\ &= \tfrac{1}{8}((2 + 2c_\angle) + (2 + 2c_\angle)A\hat{z}), \\ &= \tfrac{1+c_\angle}{4}(1 + A\hat{z}), \\ &= \tfrac{1+c_\angle}{2}\hat{R}, \end{aligned} \tag{167}$$

where there is considerable cancellation resulting from swapping the order of the canonical basis vectors $\hat{z}$ and $\hat{x}$ with A so as to get all the canonical basis vectors to the right, and then using $A\tilde{A} = 1$ to simplify. This result is the familiar $(1 + \cos(\theta))/2$ result of the Pauli algebra.

To derive $\hat{R}\hat{G}\hat{B}\hat{R}$ for the case of $B = 0$, let us begin with the standard Pauli algebra result Eq. (113), with $i = \widehat{xyz}$, and with $\hat{r} = \hat{z}$:

$$F_{rgbr} = T(rgb)e^{\phi(rgb)\widehat{xyz}}(1 + \hat{z})/2, \tag{168}$$

where $T(rgb)$ and $\phi(rgb)$ are real functions defined by the Pauli algebra and explicitly evaluated in Eq. (117) and Eq. (120). The multiplication of the canonical basis vectors on the left by A is an isomorphism of the algebra in that it preserves the additive and multiplicative relationships that define the algebra. For example:

$$\begin{aligned} A\hat{x}A\hat{y} &= A\tilde{A}\hat{x}\hat{y} = \hat{x}\hat{y}, \\ &= -\hat{y}\hat{x} = -A\tilde{A}\hat{y}\hat{x}, \\ &= -(A\hat{y})(A\hat{x}). \end{aligned} \tag{169}$$

This transformation converts the $\widehat{xyz}$ to $A\widehat{xyz}$. Alternatively, we can convert the $\widehat{xyz}$ in Eq. (168) to $\widehat{xy}$, which will then be left untouched by the transformation. To do this, note that $(1+\hat{z})$ is an eigenvector of $\hat{z}$ with eigenvalue 1. Thus we can rewrite Eq. (168) as:

$$\begin{aligned}\hat{R}\hat{G}\hat{B}\hat{R} &= T(rgb)e^{\phi(rgb)\widehat{xy}}\hat{R}, \\ &= T(rgb)e^{\phi(rgb)A\widehat{xyz}}\hat{R}.\end{aligned} \tag{170}$$

Thus we have that the situation with $B=0$ is identical to that of the usual Pauli algebra with no hidden dimension, except that we must replace the imaginary number i with either the Clifford algebraic element $\widehat{xy}$, or $A\widehat{xyz}$. Either of these constants will commute with $\hat{r}$, but only the second will necessarily also commute with $\hat{g}$ and $\hat{b}$. Both these constants square to -1, but neither must necessarily commute with all the other elements of the Clifford algebra and can therefore be considered as a geometric i.

We now let B be arbitrary and compute $\hat{R}\hat{G}\hat{R}$. Using Eq. (164), we first rewrite $\hat{G}$ as follows:

$$\begin{aligned}\hat{G} &= (1+c_\angle A\hat{z}+s_\angle A\hat{x}+B)/2, \\ &= (1+A\hat{z}+B)/2+(c_\angle-1)A\hat{z}/2+(s_\angle)A\hat{x}/2, \\ &= \hat{R}+(c_\angle-1)A\hat{z}/2+(s_\angle)A\hat{x}/2,\end{aligned} \tag{171}$$

Substituting this into $\hat{R}\hat{G}\hat{R}$ gives:

$$\hat{R}\hat{G}\hat{R}=\hat{R}+\frac{c_\angle-1}{2}\hat{R}A\hat{z}\hat{R}+\frac{s_\angle}{2}\hat{R}A\hat{x}\hat{R}, \tag{172}$$

so we need to compute $\hat{R}A\hat{z}\hat{R}$ and $\hat{R}A\hat{x}\hat{R}$.

Let us work on the last term in Eq. (172) first. We conveniently multiply by 4, multiply out terms and then factor $\hat{x}$ and $\hat{z}$ out to the right:

$$\begin{aligned}4\hat{R}A\hat{x}\hat{R} &= (1+A\hat{z}+B)A\hat{x}(1+A\hat{z}+B), \\ &= A\hat{x}+A\hat{z}A\hat{x}+BA\hat{x}+A\hat{x}A\hat{z}+A\hat{z}A\hat{x}A\hat{z} \\ &\quad +BA\hat{x}A\hat{z}+A\hat{x}B+A\hat{z}A\hat{x}B+BA\hat{x}B, \\ &= A\hat{x}-A\tilde{A}\widehat{xz}+BA\hat{x}+A\tilde{A}\widehat{xz}-A\tilde{A}A\hat{x} \\ &\quad +BA\tilde{A}\widehat{xz}+A\tilde{B}\hat{x}-A\tilde{A}B\widehat{xz}+BA\tilde{B}\hat{x}, \\ &= (A-A\tilde{A}A+BA\tilde{B}+BA+A\tilde{B})\hat{x} \\ &\quad +(BA\tilde{A}-A\tilde{A}B)\widehat{xz}.\end{aligned} \tag{173}$$

From Eq. (159), one obtains that

$$\begin{aligned}A\tilde{A} &= 1-BB, \quad \text{so} \\ BA\tilde{A} &= B-BB=A\tilde{A}B,\end{aligned} \tag{174}$$

so $BA\tilde{A}-A\tilde{A}B=0$. Similarly,

$$\begin{aligned}A\tilde{A} &= 1-BB, \quad \text{so} \\ A\tilde{A}A &= A-B(BA), \\ &= A+BA\tilde{B}, \quad \text{so} \\ 0 &= A-A\tilde{A}A+BA\tilde{B}.\end{aligned} \tag{175}$$

Substituting these relations, along with $BA+A\tilde{B}=0$ into Eq. (173) gives:

$$4\hat{R}A\hat{x}\hat{R}=(0+0)\hat{x}+(0)\widehat{xz}=0, \tag{176}$$

and the last term in Eq. (172) is zero.

The second to last term in Eq. (172) may be similarly manipluated to give:

$$\begin{aligned}4\hat{R}A\hat{z}\hat{R} &= A\hat{z}+A\tilde{A}+BA\hat{z}+A\tilde{A}+A\tilde{A}A\hat{z} \\ &\quad +BA\tilde{A}+A\tilde{B}\hat{z}+A\tilde{A}B+BA\tilde{B}\hat{z}, \\ &= (2A\tilde{A}+A\tilde{A}B+BA\tilde{A})+ \\ &\quad (A+BA+A\tilde{A}A+A\tilde{B}+BA\tilde{B})\hat{z}, \\ &= 2A\tilde{A}(1+B)+2A\tilde{A}A\hat{z}, \\ &= 2A\tilde{A}(1+A\hat{z}+B)=4A\tilde{A}\hat{R}.\end{aligned} \tag{177}$$

Upon substituting Eq. (177) and Eq. (176) into Eq. (172) we have:

$$\begin{aligned}\hat{R}\hat{G}\hat{R} &= \hat{R}+\tfrac{c_\angle-1}{2}A\tilde{A}\hat{R}, \\ &= (1-\tfrac{1-c_\angle}{2}A\tilde{A})\hat{R}.\end{aligned} \tag{178}$$

For $A\tilde{A}=1$, we have again reproduced the usual $(1+\cos(\theta_\angle))/2$ probability relationship of the Pauli algebra.

In Section (XII) we derived equations for probabilities $P_{\theta s}$ and $P_{\theta t}$ that correspond to the cases of $A=\cos(\alpha), B=\sin(\alpha)\hat{s}$ and $A=\cosh(\alpha), B=\sinh(\alpha)\hat{t}$, respectively. We can now check the results of those equations against Eq. (178) by substituting, respectively, $A\tilde{A}=\cos^2(\alpha)$ and $A\tilde{A}=\cosh^2(\alpha)$. Since we are computing here the probability associated with $\hat{R}\hat{G}\hat{R}$ instead of $\hat{R}\hat{G}$, our results are squared. For the first case, comparing against Eq. (139) we find:

$$\begin{aligned}P_{RGR} &= |\hat{R}\hat{G}\hat{R}|_G^2/|\hat{R}|_G^2, \\ &= |(1-\tfrac{1-c_\angle}{2}A\tilde{A})\hat{R}|_G^2/|\hat{R}|_G^2, \\ &= (1-\tfrac{1-c_\angle}{2}A\tilde{A})^2, \\ &= (1-\tfrac{1-c_\angle}{2}\cos^2(\alpha))^2, \\ &= (P_{\theta s})^2,\end{aligned} \tag{179}$$

as expected. For the second case, comparing against Eq. (148) we find:

$$\begin{aligned}P_{RGR} &= (1-\tfrac{1-c_\angle}{2}A\tilde{A})^2, \\ &= (1-\tfrac{1-c_\angle}{2}\cosh^2(\beta))^2, \quad \text{(bad)}\end{aligned} \tag{180}$$

which is different from Eq. (148). The problem is that our extended probability rule only computes the probability of transition between pairs of states and the above calculation has two transitions, red to green and green to red. The failure of the calculation calls into question our ability to consider products of projection operators of this sort.

The calculation for the probability of transition from red to green is, as before:

$$P_{RG}=\frac{|\hat{R}\hat{G}|_G^2}{|\hat{R}|_G^2}. \tag{181}$$

That the above gives the same result as Eq. (148) is left as an exercise for the reader. We also leave as an exercise the verification that the above calculation of $\hat{R}\hat{G}\hat{R}$, for

the case of $A\tilde{A} > 1$, gives the same result as what one obtains in making the computation with

$$\hat{u}_t = \sqrt{A\tilde{A}}\hat{u} + \sqrt{1 - A\tilde{A}}\ \hat{t} \tag{182}$$

in the manner of Section (XII).

The author has been unable to find solutions for $(A\hat{z} + B)^2 = 1$ that have $A\tilde{A}$ not real, except solutions that have $AB = BA = A\tilde{B} = 0$ such as the following:

$$A = i(\widehat{xyz} + \hat{s})/2, \qquad B = (1 - \widehat{xyzs})/2. \tag{183}$$

Nor has the author been able to prove that all solutions in which $A\tilde{A}$ is not real satisfy $AB = 0$. In the above example, A and B are in the ideals generated by the idempotents $(1+\widehat{xyzs})/2$ and $(1-\widehat{xyzs})/2$, respectively, and $A\tilde{A}$ and $B\tilde{B}$ are complementary idempotents that sum to unity. One cannot parameterize these sorts of solutions with trigonometric functions that rotate between A and B in the manner of the two examples of Section (XII).

Finally, we come to the problem of putting $\hat{R}\hat{G}\hat{B}\hat{R}$ into a form similar to Eq. (170), but with B non zero. The author's computer, programmed in Java, has explored several particular cases of this problem and has found that, subject to the limitation that $A\tilde{A}$ is a positive real number, they are always of the form:

$$\hat{R}\hat{G}\hat{B}\hat{R} = T_{RG}T_{GB}T_{BR}\ e^{\phi(rgb)\widehat{xy}}\ \hat{R}. \tag{184}$$

where $T_{\chi\chi}$ and $\phi(rgb)$ are the same real function that attends the phases of the generalized Pauli algebra. We assume that this is generally true and leave the proof as an entertainment for the reader.[8]

XIV. GEOMETRIC GENERALIZED SPINORS

The previous section considered generalizations of the Pauli algebra where $\hat{u}$ was replaced by $\hat{u}' = A\hat{u} + B$, where A and B are elements of a Clifford algebra that contains the Pauli algebra, subject to the restriction that $(A + \hat{u} + B)^2 = \hat{u}^2 = 1$. So long as A was restricted to a form where $A\tilde{A}$ is real and positive, we showed that such generalizations are equivalent to the simpler generalizations of Section (V) with corresponding phases and probabilities according to the cases:

$$\begin{array}{rcl} 0 < A\tilde{A} < 1 & \equiv & \hat{u} \rightarrow \hat{u}' = \cos(\alpha)\hat{u} + \sin(\alpha)\hat{s}, \\ 1 < A\tilde{A} & \equiv & \hat{u} \rightarrow \hat{u}' = \cosh(\beta)\hat{u} + \sinh(\beta)\hat{t}, \\ A\tilde{A} = 1 & \equiv & \hat{u} \rightarrow \hat{u}' = \hat{u}, \end{array} \tag{185}$$

where $\alpha = \beta = (A\tilde{A})^{0.5}$, and where $\hat{s}$ and $\hat{t}$ are canonical basis vectors of the Clifford algebra that square to $+1$ and -1, respectively.

[8] It smells like a general result of Lie algebra.

We now proceed to an analysis of the algebras of the $\hat{u}'$ type, with out respect to the Clifford algebra in which they are contained. That is, we will examine representations of these algebras and how they differ from the Pauli algebra while ignoring the "hidden dimension" content. We will call these "reduced" algebras in that they will contain only elements that can be generated from modified Pauli vectors rather than the whole Clifford algebra.

The correspondences listed above in Eq. (185) are true in that the computed transition probabilities between two states oriented in directions differing by an angle $\theta_\angle$ are identical, and in that there is a "phase" associated with products of three projection operators. In Eq. (170) we showed that these phases are of the form:

$$\phi(rgb)\ \widehat{xy}, \quad \text{or} \quad \phi(rgb)\ A\widehat{xyz}, \tag{186}$$

where $\phi(rgb)$ is a real number. Of these, the first, $\widehat{xy}$ takes the place of the imaginary unit in that

$$(\widehat{xy})^2 = -1, \tag{187}$$

and $A\widehat{xy}$ commutes with $\hat{r} = A\hat{z} + B$, however, $\widehat{xy}$ does not commute with $\hat{g}$ or $\hat{b}$. This is a serious defect that we will correct in the next paragraph. The second choice of imaginary unit, $A\widehat{xyz}$, also squares to -1, and commutes with any element of the form $A\hat{u}$, but does not necessarily commute with terms of the more general form $A\hat{u} + B$ unless B is real (as it is in the $\hat{u}'$ examples of Eq. (185)). We will define our spinors using the first type.

Phases appear in products of the form $\hat{R}\hat{G}\hat{B}\hat{R}$, where $\widehat{xy}$ necessarily commutes only with $\hat{R}$. This is not a problem if we restrict ourselves to only considering products that begin and end with $\hat{R}$. As we saw in Eq. (178)) and Eq. (184)), such products, subject to the requirement that $A\tilde{A}$ is real, commute with $\widehat{xy}$.

In the Pauli algebra, complex phases are associated with spherical areas by the formula $\phi = S/2$ where S is the spherical area. This is a definition that does not rely on any apparent choice of orientation. In the generalization to $\hat{u}' = A\hat{u} + B$, our phases only commute with the projection operators associated with a particular point on the sphere, $\vec{z}$, for example. To associate a phase with a spherical area in the generalized Pauli algebra, we must make a choice of this special point.[9] This is a gauge choice.

In order to arrange for our three projection operators, $\hat{R}$, $\hat{G}$ and $\hat{B}$, to be treated equally, we will choose a direction, or gauge, that is equidistant from each. In the previous section we chose $\vec{r}$, $\vec{g}$ and $\vec{b}$ so that $\vec{r}$ was oriented in the $+z$ direction. This was for calculational convenience in that we were considering products of projection operators that had more $\hat{R}$ than $\hat{G}$ or $\hat{B}$. Now,

[9] More accurately, $\widehat{xy}$ commutes with pairs of projection operators, for example $\widehat{\pm z}' = (1 \pm A\hat{z} + B)/2$. These correspond to opposite points on the sphere through the $\vec{z}$ axis.

we will instead choose coordinates so that the cone that $\vec{r}$, $\vec{g}$ and $\vec{b}$ are on is centered in the $+z$ direction. Rather than Eq. (163), we will accordingly choose:

$$\begin{array}{rcl} \vec{r} & = & (\quad s_b, \quad\quad 0, \ c_b), \\ \vec{g} & = & (\ -s_b/2, \ +s_b\sqrt{3}/2, \ c_b), \\ \vec{b} & = & (\ -s_b/2, \ -s_b\sqrt{3}/2, \ c_b), \end{array} \tag{188}$$

where c_b and s_b are the cosine and sines of θ_b, the opening angle of the cone. As before, we have that $c_\angle = c_b^2 - s_b^2/2$.

Our choice of gauge is equivalent to the selection of a vacuum state in Schwinger's measurement algebra. Since it's oriented in the z direction, we will label the projection operator in that direction $\hat{Z}$:

$$\hat{Z} = (1 + A\hat{z} + B)/2. \tag{189}$$

By the results of the previous section, we can define the phase of any triangular region that includes $\vec{z}$ as a corner by:

$$\hat{Z}\hat{U}\hat{V}\hat{Z} = T_{zu}T_{uv}T_{vz}e^{\phi(zuvz)\widehat{xy}}\ \hat{Z}, \tag{190}$$

where the other two corners are $\vec{u}$ and $\vec{v}$ and $\phi(zuvz)$ is the real value from Eq. (184)). In the above, note that $\widehat{xy}$ commutes with $\hat{Z}$, and squares to -1, so fulfills these duties of an imaginary unit, as far as the right hand side of the equation.

To define the phase of a triangular region that does not include the point $\vec{z}$, we use the additive property of phases:

$$\begin{array}{rcl} \phi(rgb) & = & \phi(zrgz) + \phi(zgbz) + \phi(zbrz), \\ & = & \phi_z(rg) + \phi_z(gb) + \phi_z(br), \end{array} \tag{191}$$

where the ϕ_z notation indicates that the ϕ function depends on the gauge choice. Because of the symmetry of our choice of gauge with respect to $\vec{r}$, $\vec{g}$ and $\vec{b}$, we have that $\phi(rgb)$ is three times the area of any of the three triangles $\phi_z(\chi\chi)$ in the above.

In keeping track of phases in this manner, we reduce our modified Pauli algebra to probabilities and phases. We have three cases, corresponding to $A\tilde{A}$ greater, equal or less than 1, but these three cases give identical phases, and as far as probabilities, both Eq. (148) and Eq. (139) are similar linear functions of $\cos(\theta)$. So long as we are concerned only with angles, phases and probabilities, we need not distinguish between the $A\tilde{A} > 1$ and $A\tilde{A} < 1$ situations so long as we arrange for their probabilities to be identical. Examining Eq. (148) and Eq. (139), we see that the two assumptions are identical provided:

$$\sin^2(\alpha) = \sinh^2(\beta), \tag{192}$$

where α is used for the positive signature and β for the negative. And both these cases include the usual Pauli algebra with $\alpha = 0$ or $\beta = 0$. Accordingly, from here on we will assume the α case with $0 \le \alpha < \pi/2$ which gives $0 < A\tilde{A} \le 1$.

FIG. 4: Table of standard model quantum numbers, weak isospin (t_3), and weak hypercharge (t_0), for the electron family.

	t_3	t_0	Q	$Q'\sqrt{3}/2$
e_R	0	-1	-1	1/2
e_L	-1/2	-1/2	-1	-1/2
ν_L	1/2	-1/2	0	1
ν_R	0	0	0	0
d_{*R}	0	-1/3	-1/3	1/6
d_{*L}	-1/2	1/6	-1/3	-5/6
u_{*L}	1/2	1/6	2/3	2/3
u_{*R}	0	2/3	2/3	-1/3

We can now define bras and kets in terms of projection operators:

$$\begin{array}{l} |R\rangle = (\hat{R}\hat{Z})/T_{RZ}, \\ \langle R| = (\hat{Z}\hat{R})/T_{RZ}. \end{array} \tag{193}$$

With this definition, the usual rules that define projection operators will function, for example:

$$\begin{array}{rcl} |R\rangle\langle R| & = & (\hat{R}\hat{Z})(\hat{Z}\hat{R})/(T_{RZ})^2, \\ & = & T_{RZ}T_{ZR}\hat{R}/(T_{RZ})^2, \\ & = & \hat{R}. \end{array} \tag{194}$$

Putting $\hat{Z}$ on the outsides of the bras and kets means that any product of projection operators that is operated upon with a bra on the left and a ket on the right gives a result that is a "complex" multiple of $\hat{Z}$. For example, to compute $\langle R|\hat{G}\hat{B}|R\rangle$, we replace $\hat{G}$ and $\hat{B}$ with their spinor products and obtain:

$$\begin{array}{rcl} & & \langle R|\ |G\rangle\langle G|\ |B\rangle\langle B|\ |R\rangle, \\ & = & (\hat{Z}\hat{R})(\hat{G}\hat{Z})(\hat{Z}\hat{G})(\hat{B}\hat{Z})(\hat{Z}\hat{B})(\hat{R}\hat{Z})/(T_{RZ}T_{GZ}T_{BZ})^2, \\ & = & (\hat{Z}\hat{R}\hat{G}\hat{Z})(\hat{Z}\hat{G}\hat{B}\hat{Z})(\hat{Z}\hat{B}\hat{R}\hat{Z})/(T_{RZ}T_{GZ}T_{BZ})^2, \\ & = & T_{RG}T_{GB}T_{BR}e^{(\phi_z(rg)+\phi_z(gb)+\phi_z(br))\hat{i}}\hat{Z}. \end{array} \tag{195}$$

where $\hat{i} = \widehat{xy}$. Upon multiplying any two objects such as the above, the Clifford algebraic parts, $\hat{i}$ and $\hat{Z}$, commute with $\hat{Z}\hat{Z} = \hat{Z}$ and $\hat{i}\hat{i} = -1$. Thus the set of such objects are equivalent to the complex numbers. They form a complex ideal of the Clifford algebra. In the application sections of this paper, we will use this form.

XV. A GEOMETRIC MODEL OF THE FERMIONS

We now examine the elementary fermions from the context of the primitive idempotents of a Clifford algebra. The primitive idempotents of a Clifford algebra arrive in hypercubic form, for example, see Eq. (53) or Eq. (81). That is, as the number of dimensions of the Clifford algebra increase, the number of independent commuting roots of unity increase and with each new root of unity, the number of primitive idempotents double. Thus we look for a hypercubic form in the elementary fermions.

FIG. 5: The fermion cube. The ν_R is not shown for clarity.

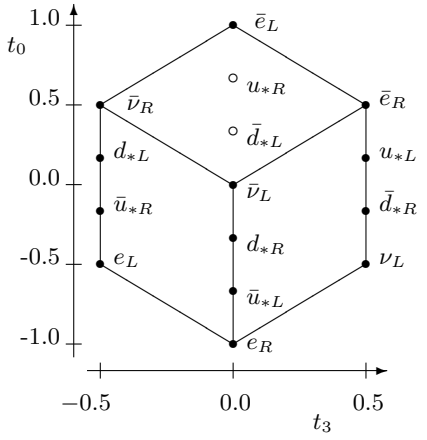
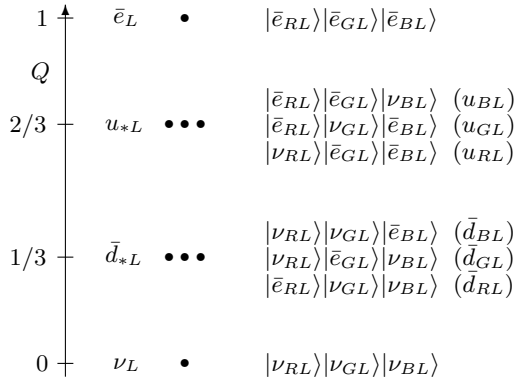


FIG. 6: A column of quarks and leptons shown as bound states of three binons each.



A table of the usual quantum numbers for the electron family fermions is shown in Fig. (4), where the right handed neutrino is needed both to obtain a number of particles that is a power of two as well as to provide a neutrino mass. The quantum numbers of the antiparticles are the negatives of those shown, and the color quantum numbers are suppressed. When the various colors and the antiparticles are included, there are a total of 32 particles. When these are plotted according to weak hypercharge and weak isospin, the result is indeed a cube, as shown in Fig. (5).

However, instead of a hypercube of dimension five that we would expect from the number of particles, (i.e. $2^5 = 32$) we have only a three dimensional cube (i.e. $2^3 = 8$). The problem is the quarks. Instead of having a power of two of quarks, they appear in 3×2^3 form, situated between an electron and an antineutrino, or between a positron and a neutrino. See Fig. (6) for a drawing of one of these structures. These columns, along with the fact that there are three families of elementary fermions, suggests that the elementary fermions are not elementary particles but are composed of subparticles or preons.

According to this preon theory, the leptons are pure states composed of three subparticles, while the quarks are mixed states composed of these same three subparticles. The three particles making up a lepton must be somehow distinguished, from the example of the quarks we will call this color. Thus the leptons, being a mixture of all three colors, are color singlets while the mixed composition of the quarks produces color triplets.

According to this theory, the preons are handed particles and therefore must be massless. To create the usual elementary particles there are two layers of condensation. The lowest layer combines three preons to produce a handed states, for example, e_L. The higher layer combines two handed states to produce an elementary fermion, for example e_L and e_R combine to produce the electron. Since the preons combine to produce the handed state before the handed states combine to produce the fermion, the force between the preons must be stronger than the force between the left and right handed fermions. This implies that the natural scale for the preon force is the Plank mass or even a multiple of it. That the elementary particles appear pointlike is natural.

XVI. THE SMA OF DEEPLY BOUND COMPOSITES

While the tools we have developed for analyzing the primitive idempotents of Clifford algebras are apparently the correct tool for analyzing the preons that we propose make up the subparticles to the handed fermions (since the preons have the cubic strucuture implied by this analysis), but the Schwinger measurement algebra was originally developed to account for properties of quantum particles in general and certainly not the speculative preons discussed here. Thus we need to derive the Schwinger measurement algebra for the case of deeply bound composite particles made up of three preons. The result of our derivation will be a 3×3 matrix whose elements are taken from a Clifford algebra that describes the preons.

In moving from the SMA for the preons to the SMA for the deeply bound composite states, we will end up representing the bound states with 3×3 matrices, with the matrices taking their elements from products of the primitive idempotents that represent the preons. In doing this, we will find that there are always three different solutions to the problem, and we will associate these three solutions to the three generations of fermions. From examining the content of the matrices, we will show that the subparticles are freely converting between one another. This suggests that there should be a matrix that corresponds to the phases and probabilities of the the various colors converting into one another. The three generations would then correspond to the three solutions to the eigenvector problem.

Let us begin with the usual (spinor) quantum theory of three (preon) particles. We will write the kets for the

three preons as $|R\rangle$, $|G\rangle$, and $|B\rangle$. Since we are assuming that the particles are deeply bound, we will assume that the wave functions are all delta functions with peaks at the same point in space, say x_0. Thus there is no position dependendence to distinguish the states. Instead, the preon states can only be distinguished by their color. This reduces the problem to a finite number of dimensions, perfect for analysis by the Schwinger measurement algebra.

Even though the combined wave function is trivial in that it is zero except at x_0, the three wave functions must still be distinguishable by their color. In the spinor form, each of the three preon states can be multiplied by an arbitrary complex constant, so we will do this explicitly (and thereby allow $|R\rangle$, $|G\rangle$, and $|B\rangle$ to be fixed constants for some gauge choice):

$$|\alpha_R, \alpha_G, \alpha_B\rangle = \begin{pmatrix} \alpha_R|R\rangle \\ \alpha_G|G\rangle \\ \alpha_B|B\rangle \end{pmatrix}. \tag{196}$$

The corresponding bra form is

$$\langle\alpha_R, \alpha_G, \alpha_B| = \begin{pmatrix} \alpha_R^*\langle R| & \alpha_G^*\langle G| & \alpha_B^*\langle B| \end{pmatrix}. \tag{197}$$

The density matrix form for the state is therefore:

$$\rho = \begin{pmatrix} |\alpha_R|^2|R\rangle\langle R| & \alpha_G^*\alpha_R|R\rangle\langle G| & \alpha_B^*\alpha_R|R\rangle\langle B| \\ \alpha_R^*\alpha_G|G\rangle\langle R| & |\alpha_G|^2|G\rangle\langle G| & \alpha_B^*\alpha_G|G\rangle\langle B| \\ \alpha_R^*\alpha_B|B\rangle\langle R| & \alpha_G^*\alpha_B|B\rangle\langle G| & |\alpha_B|^2|B\rangle\langle B| \end{pmatrix}. \tag{198}$$

We now add the requirement that R, G, and B be treated identically. We need not require that α_R, α_G and α_B be identical, since the spinor form is subject to the usual gauge symmetries, but we instead require that rotating them:

$$\begin{array}{l} \alpha_R \to \alpha_G \\ \alpha_G \to \alpha_B \\ \alpha_B \to \alpha_R \end{array} \tag{199}$$

leaves the overall density matrix unchanged. This implies that:

$$\alpha_R^*\alpha_G = \alpha_G^*\alpha_B = \alpha_B^*\alpha_R = \beta_s e^{+i\epsilon_s}, \tag{200}$$

where $\beta_s \geq 0$ and ϵ_s are real constants and the s subscript refers to the fact that these terms come from the spinor. Thus $|\alpha_R|^2 = |\alpha_G|^2 = |\alpha_B|^2 = \beta_s^2$, and the density matrix must be of the form:

$$\rho = \beta_s \begin{pmatrix} |R\rangle\langle R| & e^{-i\epsilon_s}|R\rangle\langle G| & e^{+i\epsilon_s}|R\rangle\langle B| \\ e^{+i\epsilon_s}|G\rangle\langle R| & |G\rangle\langle G| & e^{-i\epsilon_s}|G\rangle\langle B| \\ e^{-i\epsilon_s}|B\rangle\langle R| & e^{+i\epsilon_s}|B\rangle\langle G| & |B\rangle\langle B| \end{pmatrix}. \tag{201}$$

While the above, overall, is a density matrix (if $\beta_s = 1/3$), its components are written in bra ket notation. We can immediately replace the pure products with:

$$\begin{array}{l} |R\rangle\langle R| = \iota_R, \\ |G\rangle\langle G| = \iota_G, \\ |B\rangle\langle B| = \iota_B, \end{array} \tag{202}$$

where ι_χ are primitive idempotents of the underlying Clifford algebra. The cross products are more difficult as they correspond to Schwinger's general measurement symbols that we analyzed in Section (V). Such symbols imply the specification of a particular orientation gauge.

We cannot simply replace the cross symbols such as $|R\rangle\langle G|$ with $\iota_R\iota_G$ because we cannot be certain of choosing a consistent orientation gauge. Specifying the complex number $\langle R|G\rangle$ allows us to write:

$$\iota_R\, \iota_G = |R\rangle\langle R||G\rangle\langle G| = \langle R|G\rangle\, |R\rangle\langle G|, \tag{203}$$

and replace $|R\rangle\langle G|$ with $\iota_R\, \iota_G/\langle R|G\rangle$. As before, we assume that our system is symmetric under rotations of $R \to G \to B \to R$. Therefore we assume

$$\langle R|G\rangle = \langle G|B\rangle = \langle B|R\rangle = \beta_c e^{+i\epsilon_c}, \tag{204}$$

where β_c and ϵ_c are real constants and the c refers to the fact that these come from the underlying Clifford algebra.

We can now write the density matrix as a matrix with its components taken as complex multiples of the primitive idempotents associated with the preons:

$$\rho = \frac{\beta_s}{\beta_c}\begin{pmatrix} \beta_c\, \iota_R & e^{-i(\epsilon_s+\epsilon_c)}\iota_R\iota_G & e^{+i(\epsilon_s+\epsilon_c)}\iota_R\iota_B \\ e^{+i(\epsilon_s+\epsilon_c)}\iota_G\iota_R & \beta_c\, \iota_G & e^{-i(\epsilon_s+\epsilon_c)}\iota_G\iota_B \\ e^{-i(\epsilon_s+\epsilon_c)}\iota_B\iota_R & e^{+i(\epsilon_s+\epsilon_c)}\iota_B\iota_G & \beta_c\, \iota_B \end{pmatrix}. \tag{205}$$

In order to apply the restriction that ρ is idempotent, we will need to evaluate products of the ι_χ. To do this, we must use our assumption that $\langle R|G\rangle = \beta_c \exp(+i\epsilon_c)$ as follows:

$$\begin{array}{rcl} \iota_R\, \iota_G\, \iota_R &=& |R\rangle\langle R|G\rangle\langle G|R\rangle\langle R|, \\ &=& \langle R|G\rangle\, \langle G|\rangle R\, |R\rangle\langle R|, \\ &=& \beta_c^2\, \iota_R, \end{array} \tag{206}$$

similarly for $\iota_R\, \iota_B\, \iota_R$, and

$$\begin{array}{rcl} \iota_G\, \iota_B\, \iota_R &=& |G\rangle\langle G|B\rangle\langle B|R\rangle\langle R|, \\ &=& \langle G|B\rangle\, \langle B|R\rangle\, |G\rangle\langle R|, \\ &=& \frac{\langle G|B\rangle\, \langle B|R\rangle}{\langle G|R\rangle}\, |G\rangle\langle G|R\rangle\langle R|, \\ &=& \beta_c \exp(+3i\epsilon_c)\, \iota_G\, \iota_R. \end{array} \tag{207}$$

With the above, we can compute $\rho\rho$ by matrix multiplication and compare the product with ρ. The result is nine equations that all into two types. The three diagonal elements all generate equations resolve similar to:

$$\begin{array}{rcll} \beta_s\iota_R &=& \beta_s^2\iota_R\iota_R + \frac{\beta_s^2}{\beta_c^2}(\iota_R\iota_G\iota_G\iota_R + \iota_R\iota_B\iota_B\iota_R), & \\ &=& \beta_s^2\iota_R + \frac{\beta_s^2}{\beta_c^2}(\iota_R\iota_G\iota_R + \iota_R\iota_B\iota_R), & \\ &=& (\beta_s^2 + \beta_s^2 + \beta_s^2)\iota_R, & \text{so,} \\ \beta_s &=& 1/3. & \end{array} \tag{208}$$

The six off diagonal equations resolve similarly. For example:

$$\begin{array}{rcl} \frac{\beta_s}{\beta_c}e^{+i(\epsilon_s+\epsilon_c)}\iota_G\iota_R &=& \frac{2\beta_s^2}{\beta_c}e^{+i(\epsilon_s+\epsilon_c)}\iota_G\iota_R \\ && +\frac{\beta_s^2}{\beta_c^2}e^{-2i(\epsilon_s+\epsilon_c)}\iota_G\iota_B\iota_R, \end{array} \tag{209}$$

after applying Eq. (207), and substituting $\beta_s = 1/3$, this reduces to:

$$\begin{aligned} \tfrac{1}{3\beta_c} e^{+i(\epsilon_s+\epsilon_c)} &= \tfrac{2}{9\beta_c} e^{+i(\epsilon_s+\epsilon_c)} + \tfrac{1}{9\beta_c} e^{-2i(\epsilon_s)} e^{+i\epsilon_c}, \text{ or,} \\ 3 &= 2 + e^{-3i\epsilon_s} \quad \text{and so,} \\ 1 &= e^{+3i\epsilon_s}. \end{aligned} \tag{210}$$

We see that β_c and ϵ_c are not restricted, and that there are three possible values for ϵ_s:

$$\epsilon_s = \frac{2n\pi}{3}, \quad \text{with } n = 1, 2, 3, \tag{211}$$

where we have numbered these $1, 2, 3$ for convenience in keeping track of the charged leptons in order of their masses. Substituting back into the form for $|\alpha_R, \alpha_G, \alpha_B\rangle$, by choosing the gauge where the phase of $|R\rangle$ is kept zero, we have the three kets that correspond to the three generations as:

$$\begin{aligned} |1\rangle_G &= \begin{pmatrix} |R\rangle \\ e^{+2i\pi/3}|G\rangle \\ e^{-2i\pi/3}|B\rangle \end{pmatrix}, \qquad |3\rangle_G = \begin{pmatrix} |R\rangle \\ |G\rangle \\ |B\rangle \end{pmatrix}, \\ |2\rangle_G &= \begin{pmatrix} |R\rangle \\ e^{-2i\pi/3}|G\rangle \\ e^{+2i\pi/3}|B\rangle \end{pmatrix}. \end{aligned} \tag{212}$$

or more generally,

$$|n\rangle_G = \begin{pmatrix} |R\rangle \\ e^{+2in\pi/3}|G\rangle \\ e^{-2in\pi/3}|B\rangle \end{pmatrix}, \tag{213}$$

where we have given the G subscript to note that these are the geometric spinors and are not vectors over the complex numbers.

We have thus shown that the SMA, when applied to the preon model described in Section (XV), automatically produces three generations of particles. Furthermore, these three generations are described here in a single object, a 3×3 matrix of primitive idempotents (and their products). This opens the door to producing operators that are unified in that they naturally operate on particles of different generations.

From the idempotency calculation, it is clear that if Q is such an operator, and it is to have the generation kets, Eq. (212), as eigenvectors, then the operator must be able to be written in the following form:

$$Q = \begin{pmatrix} Q_{RR}\, \iota_R & Q_{RG}\, \iota_R\iota_G & Q_{RB}\, \iota_R\iota_B \\ Q_{GR}\, \iota_G\iota_R & Q_{GG}\, \iota_G & Q_{GB}\, \iota_G\iota_B \\ Q_{BR}\, \iota_B\iota_R & Q_{BG}\, \iota_B\iota_G & Q_{BB}\, \iota_B \end{pmatrix}, \tag{214}$$

where the $Q_{\chi\chi}$ are complex numbers. This method of writing an operator makes certain that the operator does not change the type of primitive idempotent in the spinor when it operates on the spinor (or the density matrix). We can further simplify the notation for this operator by getting rid of all the ι_χ notation, and instead absorb them into the $Q_{\chi\chi}$ elements. More specifically, we can apply Eq. (206) and Eq. (207) to bring the operator and eigenvector into the form of a 3×3 complex matrix and three complex vectors given by:

$$|n\rangle = \begin{pmatrix} 1 \\ e^{+2in\pi/3} \\ e^{-2in\pi/3} \end{pmatrix}. \tag{215}$$

If we are to have a theory that is symmetric under color rotations $R \to G \to B \to R$, we must apply this restriction to the form of Q. This severely reduces the freedom in defining Q:

$$\begin{aligned} Q_{RR} &= Q_{GG} = Q_{BB}, \\ Q_{RG} &= Q_{GB} = Q_{BR}, \\ Q_{RB} &= Q_{GR} = Q_{BG}. \end{aligned} \tag{216}$$

Thus the operator Q is restricted to have only three complex degrees of freedom or six real degrees of freedom. But we're not done. In addition, we require that Q have real eigenvalues. Since the sum of the diagonal elements of Q is the sum of its eigenvalues, we therefore have that Q_{RR} is real. Applying the $|3\rangle$ spinor, we next obtain that $Q_{RG} = Q^*_{GR}$. Thus there are only three real degrees of freedom left in defining Q, and we write the general cross generation operator as:

$$Q(\mu, \eta, \delta) = \mu \begin{pmatrix} 1 & \eta e^{+i\delta} & \eta e^{-i\delta} \\ \eta e^{-i\delta} & 1 & \eta e^{+i\delta} \\ \eta e^{+i\delta} & \eta e^{-i\delta} & 1 \end{pmatrix}. \tag{217}$$

Where μ, $\eta \geq 0$, and δ are real parameters. The eigenvalues of $|n\rangle$ with respect to Q are given by:

$$Q(\mu, \eta, \delta)\ |n\rangle = \mu(1 + 2\eta \cos(\delta + 2n\pi/3)), \tag{218}$$

and are real.

Since the trace of $Q = 3\mu$, the three eigenvalues sum to 3μ:

$$q_1 + q_2 + q_3 = \sum q_n = 3\mu. \tag{219}$$

On the other hand, the square of Eq. (217) has diagonal elements given by $\mu^2(1 + \eta^2)$, and we have that:

$$\sum q_n^2 = 3\mu^2(1 + 2\eta^2), \tag{220}$$

and dividing by Eq. (219) squared allows the elimination of μ as well:

$$\frac{\sum q_n^2}{(\sum q_n)^2} = \frac{1 + 2\eta^2}{3}. \tag{221}$$

At this point we haven't derived any relations on the leptons in that the number of degrees of freedom in Q (i.e. 3) is the same as the number of degrees of freedom in the eigenvalues. However, if nature is kind[10], η will prove to be something simple.

---

[10] she is

XVII. THE CHARGED FERMION MASSES

We now review the remarkable Koide mass formula. Let m_e, m_μ, and m_τ be the measured masses of the electron, muon and tau. In the standard model, these masses are arbitrary parameters, but in 1982, Koide [14] noticed that these masses approximately satisfy the relation:

$$\frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = \frac{2}{3}. \tag{222}$$

Comparing with Eq. (221), we see that is a case where $q_n = \sqrt{m_n}$ gives us $\eta = \sqrt{0.5}$. We will call this operator $\sqrt{M}$ and its eigenvalues $\sqrt{m}_n$:

$$\sqrt{M}\ |n\rangle = Q(\mu, \sqrt{0.5}, \delta)\ |n\rangle = \sqrt{m}_n\ |n\rangle. \tag{223}$$

The value of δ, which we will determine later, comes suspiciously close to the Cabbibo angle. The value of μ is simply the average of the square roots of the masses of the charged leptons. Note that the Koide relation is preserved when the square roots of the masses are negated, so we cannot here distinguish between μ and $-\mu$.

If the Koide mass relation is good, Koide has removed one degree of freedom from the standard model. We will later associate δ with the mixing angles and remove many more degrees of freedom in generalizing the Koide relationship to quarks and neutrinos.

If the relationship Eq. (222) were the result of random chance, one might suppose that improvements in the accuracy of the measurements of the masses would show a drift away from a ratio of exactly 2/3, however, time has been very kind to Koide's formula.[15] The latest Particle Data Group figures [16] give (MeV):

$$\begin{array}{rcrl} m_e &=& 0.510998918 & \pm 0.000000044 \\ m_\mu &=& 105.658369200 & \pm 0.000009400 \\ m_\tau &=& 1776.990000000 & \pm 0.290000000 \end{array} \tag{224}$$

Plugging the best estimates for the masses (i.e. $m_e = 0.510998918$, etc.), into Eq. (222), multiplying both sides by $(\sum \sqrt{m}_n)^2$ and comparing the two sides one obtains

$$\begin{array}{rcl} \text{RHS} = & \frac{2}{3}(\sqrt{0.510998918} + ...)^2 & = 1883.156, \\ \text{LHS} = & (0.510998918 + ...) & = 1883.159, \end{array} \tag{225}$$

The error, about 0.003MeV, is considerably less than the error in the mass of the tau, but the mass of the tau dominates both sides. Given a small change δ, in the mass of the tau, the RHS increases by δ, while the LHS increases by approximately

$$\begin{array}{l} \frac{d}{dm_\tau}\frac{2}{3}(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2\delta \\ = \frac{2}{3}(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})\delta/\sqrt{m_\tau}, \\ = 0.84\delta. \end{array} \tag{226}$$

Thus increasing the mass of the tau by, for example, the experimental error in its value 0.290MeV, should increase the LHS of the Koide formula by $0.84 \times 0.290 = 0.244$, about 0.046 less than the increase to the RHS. This would result in an error more than ten times greater than the observed error in Eq. (225). One supposes that the experimentalists have been careful with their error bars.

Fermions that are identical except for generation have masses that renormalize proportionately.[17] Thus the ratios of the observed masses of the neutrinos are proportional to the bare neutrino masses. This does not apply to two fermions of different types such as a neutrino and quark. Consequently, we can extract little use in the values of the μ parameter of $Q(\mu, \eta, \delta)$, and we will generally ignore it from here on.

We've been using $|n\rangle$ to indicate the ket for the nth generation charged lepton. We need to generalize our notation for convenient analysis of the quarks and neutrinos. Accordingly, let us follow the notation of the neutrinos, and describe these particles with a digit for the generation, and a letter to indicate the particle type. Thus the electron, muon, and tau are indicated as $|e1\rangle$, $|e2\rangle$, and $|e3\rangle$; the up quark, charm quark, and top quark are $|u1\rangle$, $|u2\rangle$, and $|u3\rangle$; the down quark, strange quark and bottom quark are $|d1\rangle$, $|d2\rangle$, and $|d3\rangle$; and the three neutrinos (as mass eigenstates) are $|\nu1\rangle$, $|\nu2\rangle$, and $|\nu3\rangle$. With this notation, our fermion zoo becomes:

$$\begin{array}{llll} |e1\rangle & |e2\rangle & |e3\rangle & \text{electron, muon, tau,} \\ |u1\rangle & |u2\rangle & |u3\rangle & \text{up, charm, top,} \\ |d1\rangle & |d2\rangle & |d3\rangle & \text{down, strange, bottom,} \\ |\nu1\rangle & |\nu2\rangle & |\nu3\rangle & \nu_1,\ \nu_2,\ \nu_3, \end{array} \tag{227}$$

If we wish to refer to the nth particle in a generation, we will use, for example, $|en\rangle$.

XVIII. FAMILY MIXING ANGLES

This section discusses the fermion mixing angles in the context of the preon model discussed above.

In this paper we've been using the density matrix formalism for pure states only. Since we are considering deeply bound states, it is natural for us to treat our quark states in the density matrix formalism as impure states. Thus we could assume that the density matrix for an up quark will be of the form:

$$\rho_{u1} = \frac{2}{3}\rho_{\bar{e}1} + \frac{1}{3}\rho_{\nu 1}\ \text{(bad)}. \tag{228}$$

However, in the above, the density matrices for the two leptons are for distinct particles. That is, they correspond to primitive idempotents that multiply to zero. So when ρ_{up} is squared, one will not obtain ρ_{up}, but will instead get

$$(\rho_{u1})^2 = \frac{4}{9}\rho_{\bar{e}1} + \frac{1}{9}\rho_{\nu 1}\ \text{(bad)} \tag{229}$$

Instead of the fractions 2/3 and 1/3, if we wish to have the idempotency relation we would have to use coefficients of 1. This would correspond to a physical situation

of two free particles, one a neutrino, the other a charged lepton.

A solution is to treat the $\bar{e}$ and ν preons as identical particles, from the point of view of the density matrix that represents their combined wave function (and therefore the binding force between them). Thus we can reuse the work of Section (XVI), as far as the wave states for the preons. We then need two square root mass operators, one for the charged leptons, the other for the neutrinos:

$$\begin{array}{rcll} \sqrt{M}_e\,|en\rangle &=& \sqrt{m}_{en}\,|en\rangle, & \text{charged leptons,} \\ \sqrt{M}_\nu\,|\nu n\rangle &=& \sqrt{m}_{\nu n}\,|\nu n\rangle, & \text{neutrinos.} \end{array} \tag{230}$$

XIX. QUARK AND NEUTRINO MASSES

This section discusses the neutrino masses as predicted by the quark and lepton square root mass operators.

We speculate that the (bare) square root masses of the quarks is

$$\begin{array}{rcl} \sqrt{m}_{un} &=& (2\sqrt{m}_{en} + \sqrt{m}_{\nu n})/3, \\ \sqrt{m}_{dn} &=& (\sqrt{m}_{en} + 2\sqrt{m}_{\nu n})/3. \end{array} \tag{231}$$

In the above, it should be noted that we cannot be sure which square roots are positive and which are negative.

XX. CONCLUSIONS

XXI. ACKNOWLEDGEMENTS

Whenas in silks my Julia goes,
Then, then, methinks, how sweetly flows
The liquefaction of her clothes.

Next, when I cast mine eyes, and see
That brave vibration, each way free,
O, how that glittering taketh me![11]

The author would like to thank Mark Mollo, owner of Liquafaction Corp., for his financial support for this work. Thanks are also due to professional physicists who have encouraged the author, most particularly L. P. Horwitz.

APPENDIX A: PAULI ALGEBRA PROBABILITY CALCULATION

Suppose two spin$-1/2$ states, α and β are represented by two normalized spinors:

$$|\alpha\rangle = \frac{1}{\sqrt{1+|\alpha|^2}}\begin{pmatrix} 1 \\ \alpha \end{pmatrix}, \qquad |\beta\rangle = \frac{1}{\sqrt{1+|\beta|^2}}\begin{pmatrix} 1 \\ \beta \end{pmatrix}. \tag{A1}$$

Then the probability of a transition between these spinors is given by the usual spinor formula, which can be manipulated into density matrix form:

$$\begin{array}{rcl} P_{\alpha\beta} &=& \langle\beta|\alpha\rangle\langle\alpha|\beta\rangle, \\ &=& \mathrm{tr}(|\beta\rangle\langle\beta||\alpha\rangle\langle\alpha|), \\ &=& \mathrm{tr}(\rho_\beta\rho_\alpha), \\ &=& \frac{1}{1+|\beta|^2}\frac{1}{1+|\alpha|^2}\mathrm{tr}(\begin{pmatrix} 1+\alpha^*\beta & \beta^*+\alpha^*|\beta|^2 \\ \alpha+\beta|\alpha|^2 & \beta^*\alpha+|\beta|^2 \end{pmatrix}, \\ &=& \frac{1+\alpha^*\beta+\beta^*\alpha+|\alpha\beta|^2}{(1+|\alpha|^2)(1+|\beta|^2)}. \end{array} \tag{A2}$$

We now search for a method of obtaining this same result in geometric form, without using the trace function. We will work on the form of the density matrix equation for probability, that is, $\mathrm{tr}(\rho_\beta\rho_\alpha)$.

It is possible to write the complex 2×2 matrices as a real vector space over products of Pauli spin matrices. There are eight real degrees of freedom in complex 2×2 matrices, so there are eight basis elements for the vector space:

$$\begin{array}{cccc} \hat{1} & \sigma_x & \sigma_y & \sigma_z \\ \sigma_x\sigma_y\sigma_z & \sigma_y\sigma_z & \sigma_x\sigma_z & \sigma_x\sigma_y, \end{array} \tag{A3}$$

where $\hat{1}$ is the unit matrix. Any complex 2×2 matrix can be written as a sum of real multiples of these basis elements. For example:

$$\begin{array}{l} \begin{pmatrix} 7-6i & 10+3i \\ -6-3i & -1-4i \end{pmatrix}, \\ = \begin{pmatrix} 3+4-i-5i & 2+3i+8-0i \\ 2-3i-8+0i & 3-4+i-5i \end{pmatrix} \\ = 3\hat{1}+2\sigma_x-3\sigma_y+4\sigma_z+\sigma_x\sigma_y+8\sigma_x\sigma_z+0\sigma_y\sigma_z-5\sigma_x\sigma_y\sigma_z. \end{array} \tag{A4}$$

This defines the 2×2 complex matrices in entirely real, and geometric, terms. That is, $\hat{1}$ is a Clifford algebra scalar, σ_x, σ_y and σ_z are Clifford algebra vectors, the products are bivectors (or pseudo vectors), and $\sigma_x\sigma_y\sigma_z \equiv i$ is the pseudoscalar.

Given a real vector space, a natural squared magnitude can be defined on the vector space by summing the squares of the real numbers that define an element according to the given basis. For this we will write $|\ \ |^2_G$, where the G subscript stands for geometric. In the above example:

$$\begin{array}{l} \left|\begin{pmatrix} 3+4-i-5i & 2+3i+8-0i \\ 2-3i-8+0i & 3-4+i-5i \end{pmatrix}\right|^2_G \\ = 3^2+2^2+(-3)^2+4^2+1^2+8^2+0^2+5^2. \end{array} \tag{A5}$$

It is remarkable that the above squared magnitude, when applied to the matrix $\rho_\beta\rho_\alpha$, gives a result that is exactly half of $P_{\alpha\beta} = \mathrm{tr}(\rho_\beta\rho_\alpha)$. The factor of two comes from the fact that $|\hat{1}|^2_G = 1$, while $\mathrm{tr}(\hat{1}) = 2$. Other than the factor of two, the calculations are identical.

To show that $\mathrm{tr}(\rho_\beta\rho_\alpha) = 2|\rho_\beta\rho_\alpha|^2_G$, one must wade through a certain amount of algebra. It is easier to do this if we transform from the purely geometric real vector

[11] "Upon Julia's Clothes", by Robert Herrick, 1591-1674

space, to a more convenient complex vector space. To do this, we make the substitution $\sigma_x\sigma_y\sigma_z = i$ and use as our complex basis:

$$\hat{1} \qquad \sigma_x \qquad \sigma_y \qquad \sigma_z. \tag{A6}$$

It is easy to see that the complex basis gives the same $|\ |^2_G$ as the real basis.

First, let us abbreviate the multiplication constant as $\kappa = 1/(1+|\alpha|^2)(1+|\beta|^2)$ to give:

$$\rho_\beta\rho_\alpha = \kappa \begin{pmatrix} 1+\alpha^*\beta & \beta^*+\alpha^*|\beta|^2 \\ \alpha+\beta|\alpha|^2 & \beta^*\alpha+|\beta|^2 \end{pmatrix}. \tag{A7}$$

We write $\rho_\beta\rho_\alpha$ in the complex basis as follows:

$$\rho_\beta\rho_\alpha = c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + c_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A8}$$

with complex coefficients c_χ. Comparing with Eq. (A7), we solve for c_χ to obtain:

$$\begin{array}{rcl} c_1 &=& \kappa(1+\alpha^*\beta+\beta^*\alpha+|\alpha\beta|^2)/2, \\ c_x &=& \kappa(\alpha+\beta|\alpha|^2+\beta^*+\alpha^*|\beta|^2)/2, \\ c_y &=& \kappa(\alpha+\beta|\alpha|^2-\beta^*-\alpha^*|\beta|^2)/2i, \\ c_z &=& \kappa(1+\alpha^*\beta-\beta^*\alpha-|\alpha\beta|^2)/2. \end{array} \tag{A9}$$

Multiplying by the complex conjugates gives:

$$\begin{array}{rcl} |c_1|^2 &=& \kappa^2(1+(\beta^*\alpha)^2+(\alpha^*\beta)^2+|\alpha\beta|^4++2\alpha^*\beta \\ &+& 2\beta^*\alpha+\beta^*\alpha|\alpha\beta|^2+\alpha^*\beta|\alpha\beta|^2+4|\alpha\beta|^2)/4, \\ |c_x|^2 &=& \kappa^2(|\alpha|^2+|\beta|^2+|\alpha|^2|\beta|^4+|\beta|^2|\alpha|^4+\alpha^2|\beta|^2 \\ &+& \beta^2|\alpha|^2+\alpha\beta+\alpha^*\beta^*+\beta*\alpha|\alpha|^2+\beta^*\alpha|\beta|^2 \\ &+& \alpha^*\beta|\alpha|^2+\alpha^*\beta|\beta|^2+\alpha\beta|\alpha\beta|^2+\alpha^*\beta^*|\alpha\beta|^2 \\ &+& (\beta^*)^2|\alpha|^2+(\alpha^*)^2|\beta|^2)/4, \\ |c_y|^2 &=& \kappa^2(|\alpha|^2+|\beta|^2+|\alpha|^2|\beta|^4+|\beta|^2|\alpha|^4-\alpha^2|\beta|^2 \\ &-& \beta^2|\alpha|^2-\alpha\beta-\alpha^*\beta^*+\beta*\alpha|\alpha|^2+\beta^*\alpha|\beta|^2 \\ &+& \alpha^*\beta|\alpha|^2+\alpha^*\beta|\beta|^2-\alpha\beta|\alpha\beta|^2-\alpha^*\beta^*|\alpha\beta|^2 \\ &-& (\beta^*)^2|\alpha|^2-(\alpha^*)^2|\beta|^2)/4, \\ |c_z|^2 &=& \kappa^2(1+(\beta^*\alpha)^2+(\alpha^*\beta)^2+|\alpha\beta|^4-(\alpha^*\beta)^2 \\ &-& (\beta^*\alpha)^2)/4. \end{array} \tag{A10}$$

The above four terms simplify when they are summed:

$$\begin{array}{rcl} |\rho_\alpha\rho_\beta|^2_G &=& |c_1|^2+|c_x|^2+|c_y|^2+|c_z|^2, \\ &=& \kappa^2(1+|\alpha|^2+|\beta|^2+|\alpha\beta|^2+ \\ &+& \alpha^*\beta+\alpha^*\beta|\alpha|^2+\alpha^*\beta|\beta|^2+\alpha^*\beta|\alpha\beta|^2 \\ &+& \beta^*\alpha+\beta^*\alpha|\alpha|^2+\beta^*\alpha|\beta|^2+\beta^*\alpha|\alpha\beta|^2 \\ &+& |\alpha\beta|^2+|\alpha|^2|\alpha\beta|^2+|\beta|^2|\alpha\beta|^2+|\alpha\beta|^4)/2, \\ &=& \kappa^2(1+|alpha|^2)(1+|\beta|^2) \\ && (1+\alpha^*\beta+\beta^*\alpha+|\alpha\beta|^2)/2, \\ &=& \frac{1+\alpha^*\beta+\beta^*\alpha+|\alpha\beta|^2}{2(1+|\alpha|^2)(1+|\beta|^2)}. \end{array} \tag{A11}$$

Comparing with Eq. (A2), we see that

$$P_{\alpha\beta} = \mathrm{tr}(\rho_\beta\rho_\alpha) = 2|\rho_\beta\rho_\alpha|^2_G. \tag{A12}$$

as desired. Pure density matrices defined by spinors chosen from any of the usual 4×4 complex matrix representations of the Dirac algebra have a similar property, but with a factor of 4 rather than 2.

APPENDIX B: ASTROPHYSICAL EVIDENCE

There are a number of odd observations in astrophysics that can be interpreted as evidence for superluminal subparticles. In this section we assemble this evidence, such as it is. Of course "a black-hole a priori can be the source of tachyonic matter,"[18] and since black holes are known to be the source of very energetic jets of particles, it is natural to suppose that the engines of these jets are tachyons.

First, the cosmic ray observations. Cosmic rays create particle showers. If the primary particle is travelling at close to the speed of light, as is the assumed case for very high energy particles, the shower particles will travel at close to the same speed and the assembly will arrive on the ground as close to a single pulse.

On the other hand, if the primary particle travels at faster than light, and if it survives for more than just one collision, then it is possible that the resulting shower, instead of consisting of a single pulse, will instead arrive as a series of pulses. This could be detected in two ways, providing the primary particle lasts long enough.

First, the resulting particle shower will be extended in time. Most of the cosmic ray experiments are not sensitive to the exact (i.e. to 10 or 100ns accuracy) arrival time of the shower particles. This is due to the fact that showers naturally broaden in time, and historically, there has been little theoretical reason to expect anomalously extended particle showers. The design of a few cosmic ray experiments, in particular AGASA, take into account the assumption that the showers are of short duration. This is a design weakness that leaves these sorts of cosmic ray detectors subject to spoofing by extended particle showers.

In the energy measurements of ultra high energy cosmic rays (UHECRs), one experiment, AGASA has produced results that suggest that their equipment provides anomalously high energies at the high end of the spectrum. The designers of AGASA measure particle energy by measuring the amount of Cherenkov light emitted by shower particles that traverse tanks of water. Electronically, the pulse is allowed to decay exponentially through an RC circuit, and the amount of energy is measured as the length of time the circuit stays above a fixed voltage.

This sort of circuit gives a logarithmic estimate of the energy and naturally gives a large dynamic range for measurements. However, such a circuit is subject to spoofing if late pulses arrive. That this will result in a too energy measurement is well known, for example, see Drescher and Farrar's article on the effect, [19].

The spoofing at AGASA only applies to tachyonic particles, and because the theory in this paper implies that color is not conserved, we would expect that our tachyons could not travel great distances. Accordingly, one might suppose that the spoofs at AGASA would be correlated with relatively short distance sources of tachyons. This is consistent with the fact that evidence for correlations between AGN and UHECR are seen only in the AGASA

data. For example, see [20, 21]

The second effect of tachyonic particles is that they imply a preferred reference frame. This is a modification of Einstein's special relativity. The appropriate theory is Lorentz's relativity sometimes called "Lorentzian Ether Theory" or LET, the theory from which Einstein borrowed the mathematics. To modify Einstein's relativity to LET, one simply adds the assumption of a preferred frame of reference. It is in that preferred reference frame that the speeds of the tachyons are defined.

In the preferred frame of reference of the LET, a typical cosmic ray target has some relatively small velocity. As long as cosmic rays are generated by particles travelling at close to light speed, this small velocity has little effect on the tracks left by the cosmic ray. In the case of a tachyonic primary particle, however, the track left by the primary particle and the track left by the secondary showers diverge. If the target is moving perpendicular to the path of the secondary, the result will be that the primary particle will leave a track at a slight angle with respect to the secondaries.

The secondary tracks will be parallel to each other, but the primary particle will form an angle with respect to these tracks. This will leave anomalous effects in the taret. First, the secondary tracks will be aligned in the direction in which the target is moving with respect to the preferred frame of reference. Second, if the primary particle leaves a track through the target, the small angle between it and the secondary tracks will be interpreted by physicists as a very high transverse momenta. There are only a few experiments that would be sensitive to this sort of anomaly. They consist of layers of x-ray emulsion placed between plates of lead, and these experiments do detect such unusual alignments and high transverse momenta.[22].

One of the odd features of modern cosmology is the assumption of an inflationary period between the big bang and the emergence of modern galaxies. The inflation is required in order to allow distant parts of the universe to come into thermal equilibrium. A tachyonic particle that condenses into standard matter provides exactly the mechanism needed to provide this inflation, as has been speculated widely among cosmologists.

Speculation that not all particles have c as a maximum speed is provided by Glashow et al, as a mechanism for neutrino mixing in [23].

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