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THE ALGEBRA OF MICROSCOPIC MEASUREMENT

BY JULIAN SCHWINGER

HARVARD UNIVERSITY

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This note initiates a brief account of the fundamental mathematical structure of quantum mechanics, not as an independent mathematical discipline with physical applications, but evolved naturally as the symbolic expression of the physical laws that govern the microscopic realm.[1]

The classical theory of measurement is implicitly based upon the concept of an interaction between the system of interest and the measurement apparatus that can be made arbitrarily small, or at least precisely compensated, so that one can speak meaningfully of an idealized measurement that disturbs no property of the system. The classical representation of physical quantities by numbers is the identification of all properties with the results of such nondisturbing measurements. It is characteristic of atomic phenomena, however, that the interaction between system and instrument cannot be indefinitely weakened. Nor can the disturbance produced by the interaction be compensated since it is only statistically predictable. Accordingly, a measurement of one property can produce uncontrollable changes in the value previously assigned to another property, and it is without meaning to ascribe numerical values to all the attributes of a microscopic system. The mathematical language that is appropriate to the atomic domain is found in the symbolic transcription of the laws of microscopic measurement.

The basic concepts are developed most simply in the context of idealized physical systems which are such that any physical quantity A assumes only a finite number of distinct values, $a', \ldots a^n$. In the most elementary type of measurement, an ensemble of independent similar systems is sorted by the apparatus into subensembles, distinguished by definite values of the physical quantity being measured. Let $M(a')$ symbolize the selective measurement that accepts systems possessing the value a' of property A and rejects all others. We define the addition of such symbols to signify less specific selective measurements that produce a subensemble associated with any of the values in the summation, none of these being distinguished by the measurement. The multiplication of the measurement symbols represents the successive performance of measurements (read from right to left). It follows from the physical meaning of these operations that addition is commutative and associative, while multiplication is associative. With 1 and 0 symbolizing the measurements that, respectively, accept and reject all systems, the properties of the elementary selective measurements are expressed by

$$M(a')M(a') = M(a') \tag{1}$$

$$M(a')M(a'') = 0, \qquad a' \neq a'' \tag{2}$$

$$\sum_{a'} M(a') = 1. \tag{3}$$

Indeed, the measurement symbolized by $M(a')$ accepts every system produced by $M(a')$ and rejects every system produced by $M(a'')$, $a'' \neq a'$, while a selective measurement that does not distinguish any of the possible values of a' is the measurement that accepts all systems.

According to the significance of the measurements denoted as 1 and 0, these symbols have the algebraic properties

$$\begin{gathered} 1\,1 = 1, \qquad 0\,0 = 0 \\ 1\,0 = 0\,1 = 0 \\ 1 + 0 = 1, \end{gathered}$$

and

$$\begin{gathered} 1M(a') = M(a')1 = M(a'), \, 0M(a') = M(a')0 = 0 \\ M(a') + 0 = M(a'), \end{gathered}$$

which justifies the notation. The various properties of 0, $M(a')$, and 1 are consistent, provided multiplication is distributive. Thus,

$$\sum_{a''} M(a')M(a'') = M(a') = M(a')1 = M(a')\sum_{a''} M(a'').$$

The introduction of the numbers 1 and 0 as multipliers, with evident definitions, permits the multiplication laws of measurement symbols to be combined in the single statement

$$M(a')M(a'') = \delta(a',a'')M(a'),$$

where

$$\delta(a',a'') = \begin{cases} 1, & a' = a'' \\ 0, & a' \neq a''. \end{cases}$$

Two physical quantities A_1 and A_2 are said to be compatible when the measurement of one does not destroy the knowledge gained by prior measurement of the other. The selective measurements $M(a_1')$ and $M(a_2')$, performed in either order, produce an ensemble of systems for which one can simultaneously assign the values a_1' to A_1 and a_2' to A_2. The symbol of this compound measurement is

$$M(a_1'a_2') = M(a_1')M(a_2') = M(a_2')M(a_1').$$

By a complete set of compatible physical quantities, $A_1, \ldots A_k$, we mean that every pair of these quantities is compatible and that no other quantities exist, apart from functions of the set A, that are compatible with every member of this set. The measurement symbol

$$M(a') = \prod_{i=1}^{k} M(a_i')$$

then describes a complete measurement, which is such that the systems chosen possess definite values for the maximum number of attributes; any attempt to determine the value of still another independent physical quantity will produce uncontrollable changes in one or more of the previously assigned values. Thus the optimum state of knowledge concerning a given system is realized by subjecting it to a complete selective measurement. The systems admitted by the complete measurement $M(a')$ are said to be in the state a'. The symbolic properties of complete measurements are also given by equations (1), (2), and (3).

A more general type of measurement incorporates a disturbance that produces a change of state. (It is here that we go beyond previous developments along these lines.) The symbol $M(a',a'')$ indicates a selective measurement in which systems are accepted only in the state a'' and emerge in the state a'. The measurement process $M(a')$ is the special case for which no change of state occurs,

$$M(a') = M(a',a').$$

The properties of successive measurements of the type $M(a',a'')$ are symbolized by

$$M(a',a'')M(a''',a^{\text{iv}}) = \delta(a'',a''')M(a',a^{\text{iv}}), \tag{4}$$

for, if $a'' \neq a'''$, the second stage of the compound apparatus accepts none of the systems that emerge from the first stage, while if $a'' = a'''$, all such systems enter the second stage and the compound measurement serves to select systems in the state a^{iv} and produce them in the state a'. Note that if the two stages are reversed, we have

$$M(a''',a^{\text{iv}})M(a',a'') = \delta(a',a^{\text{iv}})M(a''',a''),$$

which differs in general from equation (4). Hence the multiplication of measurement symbols is noncommutative.

The physical quantities contained in one complete set A do not comprise the totality of physical attributes of the system. One can form other complete sets, B, C, . . ., which are mutually incompatible, and for each choice of noninterfering physical characteristics there is a set of selective measurements referring to systems in the appropriate states, $M(b',b''), M(c',c'')$ . . . . The most general selective measurement involves two incompatible sets of properties. We symbolize by $M(a',b')$ the measurement process that rejects all impinging systems except those in the state b', and permits only systems in the state a' to emerge from the apparatus. The compound measurement $M(a',b')M(c',d')$ serves to select systems in the state d' and produce them in the state a', which is a selective measurement of the type $M(a',d')$. But, in addition, the first stage supplies systems in the state c' while the second stage accepts only systems in the state b'. The examples of compound measurements that we have already considered involve the passage of all systems or no systems between the two stages, as represented by the multiplicative numbers 1 or 0. More generally, measurements of properties B, performed on a system in a state c' that refers to properties incompatible with B, will yield a statistical distribution of the possible values. Hence, only a determinate fraction of the systems emerging from the first stage will be accepted by the second stage. We express this by the general multiplication law

$$M(a',b')M(c',d') = \langle b'|c'\rangle M(a',d'), \tag{5}$$

where $\langle b'|c'\rangle$ is a number characterizing the statistical relation between the states b' and c'. In particular,

$$\langle a'|a''\rangle = \delta(a',a'').$$

Special examples of (5) are

$$M(a')M(b',c') = \langle a'|b'\rangle M(a',c')$$

and

$$M(a',b')M(c') = \langle b'|c'\rangle M(a',c').$$

We infer from the fundamental measurement symbol property (3) that

$$\begin{aligned}\sum_{a'}\langle a'|b'\rangle M(a',c') &= \sum_{a'} M(a')M(b',c') \\ &= M(b',c')\end{aligned}$$

and similarly

$$\sum_{c'}\langle b'|c'\rangle M(a',c') = M(a',b'),$$

which shows that measurement symbols of one type can be expressed as a linear combination of the measurement symbols of another type. The general relation is

$$\begin{aligned}M(c',d') &= \sum_{a'b'} M(a')M(c',d')M(b') \\ &= \sum_{a'b'}\langle a'|c'\rangle\langle d'|b'\rangle M(a',b').\end{aligned} \tag{6}$$

From its role in effecting such connections, the totality of numbers $\langle a'|b'\rangle$ is called the transformation function relating the a- and b-descriptions, where the phrase "a-description" signifies the description of a system in terms of the states produced by selective measurements of the complete set of compatible physical quantities A.

A fundamental composition property of transformation functions is obtained on comparing

$$\sum_{b'} M(a')M(b')M(c') = \sum_{b'}\langle a'|b'\rangle\langle b'|c'\rangle M(a',c')$$

with

$$M(a')(\sum_{b'} M(b'))M(c') = M(a')M(c')$$
$$= \langle a'|c'\rangle M(a',c'),$$

namely

$$\sum_{b'} \langle a'|b'\rangle\langle b'|c'\rangle = \langle a'|c'\rangle.$$

On identifying the a- and c-descriptions this becomes

$$\sum_{b'} \langle a'|b'\rangle\langle b'|a''\rangle = \delta(a',a'')$$

and similarly

$$\sum_{a'} \langle b'|a'\rangle\langle a'|b''\rangle = \delta(b',b'').$$

As a consequence, we observe that

$$\sum_{a'}\sum_{b'} \langle a'|b'\rangle\langle b'|a'\rangle = \sum_{a'} 1$$
$$= \sum_{b'}\sum_{a'} \langle b'|a'\rangle\langle a'|b'\rangle = \sum_{b'} 1,$$

which means that N, the total number of states obtained in a complete measurement, is independent of the particular choice of compatible physical quantities that are measured. Hence the total number of measurement symbols of any specified type is N^2. Arbitrary numerical multiples of measurement symbols in additive combination thus form the elements of a linear algebra of dimensionality N^2—the algebra of measurement. The elements of the measurement algebra are called operators.

The number $\langle a'|b'\rangle$ can be regarded as a linear numerical function of the operator $M(b',a')$. We call this linear correspondence between operators and numbers the trace,

$$\langle a'|b'\rangle = trM(b',a'), \tag{7}$$

and observe from the general linear relation (6) that

$$trM(c',d') = \sum_{a'b'} \langle a'|c'\rangle\langle d'|b'\rangle trM(a',b')$$
$$= \sum_{a'b'} \langle d'|b'\rangle\langle b'|a'\rangle\langle a'|c'\rangle$$
$$= \langle d'|c'\rangle,$$

which verifies the consistency of the definition (7). In particular,

$$trM(a',a'') = \delta(a',a'')$$
$$trM(a') = 1.$$

The trace of a measurement symbol product is

$$\begin{aligned} trM(a',b')M(c',d') &= \langle b'|c'\rangle trM(a',d') \\ &= \langle b'|c'\rangle\langle d'|a'\rangle, \end{aligned} \tag{8}$$

which can be compared with

$$trM(c',d')M(a',b') = \langle d'|a'\rangle trM(c',b') = \langle d'|a'\rangle\langle b'|c'\rangle.$$

Hence, despite the noncommutativity of multiplication, the trace of a product of two factors is independent of the multiplication order. This applies to any two elements X, Y, of the measurement algebra,

$$trXY = trYX.$$

A special example of (8) is

$$trM(a')M(b') = \langle a'|b'\rangle\langle b'|a'\rangle \tag{9}$$

It should be observed that the general multiplication law and the definition of the trace are preserved if we make the substitutions

$$\begin{aligned} M(a',b') &\rightarrow \lambda(a')^{-1}M(a',b')\lambda(b') \\ \langle a'|b'\rangle &\rightarrow \lambda(a')\langle a'|b'\rangle\lambda(b')^{-1}, \end{aligned} \tag{10}$$

where the numbers $\lambda(a')$ and $\lambda(b')$ can be given arbitrary nonzero values. The elementary measurement symbols $M(a')$ and the transformation function $\langle a'|a''\rangle$ are left unaltered. In view of this arbitrariness, a transformation function $\langle a'|b'\rangle$ cannot, of itself, possess a direct physical interpretation but must enter in some combination that remains invariant under the substitution (10).

The appropriate basis for the statistical interpretation of the transformation function can be inferred by a consideration of the sequence of selective measurements $M(b')M(a')M(b')$, which differs from $M(b')$ in virtue of the disturbance attendant upon the intermediate A-measurement. Only a fraction of the systems selected in the initial B-measurement is transmitted through the complete apparatus. Correspondingly, we have the symbolic equation

$$M(b')M(a')M(b') = p(a',b')M(b'),$$

where the number

$$p(a',b') = \langle a'|b'\rangle\langle b'|a'\rangle \tag{11}$$

is invariant under the transformation (10). If we perform an A-measurement that does not distinguish between two (or more) states, there is a related additivity of the numbers $p(a',b')$,

$$M(b')(M(a') + M(a''))M(b') = (p(a',b') + p(a'',b'))M(b)',$$

and, for the A-measurement that does not distinguish among any of the states, there appears

$$M(b')(\sum_{a'}M(a'))M(b') = M(b'),$$

whence

$$\sum_{a'}p(a',b') = 1.$$

These properties qualify $p(a',b')$ for the role of the probability that one observes the state a' in a measurement performed on a system known to be in the state b'.

But a probability is a real, nonnegative number. Hence we shall impose an admissible restriction on the numbers appearing in the measurement algebra, by requiring that $\langle a' | b' \rangle$ and $\langle b' | a' \rangle$ form a pair of complex conjugate numbers.[2]

$$\langle b' | a' \rangle = \langle a' | b' \rangle^*, \tag{12}$$

for then

$$p(a',b') = |\langle a' | b' \rangle|^2 \geq 0.$$

To maintain the complex conjugate relation (12), the numbers $\lambda(a')$ of (10) must obey

$$\lambda(a')^* = \lambda(a')^{-1},$$

and therefore have the form

$$\lambda(a') = e^{i\varphi(a')}$$

in which the phases $\varphi(a')$ can assume arbitrary real values.

Another satisfactory aspect of the probability formula (11) is the symmetry property

$$p(a',b') = p(b',a').$$

Let us recall the arbitrary convention that accompanies the interpretation of the measurement symbols and their products—the order of events is read from right to left (sinistrally). But any measurement symbol equation is equally valid if interpreted in the opposite sense (dextrally), and no physical result should depend upon which convention is employed. On introducing the dextral interpretation, $\langle a' | b' \rangle$ acquires the meaning possessed by $\langle b' | a' \rangle$ with the sinistral convention. We conclude that the probability connecting states a' and b' in given sequence must be constructed symmetrically from $\langle a' | b' \rangle$ and $\langle b' | a' \rangle$. The introduction of the opposite convention for measurement symbols will be termed the adjoint operation, and is indicated by†. Thus,

$$M(a',b')\dagger = M(b',a')$$

and

$$M(a',a'')\dagger = M(a'',a').$$

In particular,

$$M(a')\dagger = M(a'),$$

which characterizes $M(a')$ as a self-adjoint or Hermitian operator. For measurement symbol products we have

$$\begin{aligned}(M(a',b')M(c',d'))\dagger &= M(d',c')M(b',a') \\ &= M(c',d')\dagger M(a',b')\dagger,\end{aligned}$$

or equivalently,

$$\begin{aligned}(\langle b' | c' \rangle M(a',d'))\dagger &= \langle c' | b' \rangle M(d',a') \\ &= \langle b' | c' \rangle^* M(a',d')\dagger.\end{aligned}$$

The significance of addition is uninfluenced by the adjoint procedure, which permits us to extend these properties to all elements of the measurement algebra:

$$(X + Y)\dagger = X\dagger + Y\dagger,\ (XY)\dagger = Y\dagger X\dagger,\ (\lambda X)\dagger = \lambda^* X\dagger,$$

in which λ is an arbitrary number.

The use of complex numbers in the measurement algebra implies the existence of a dual algebra in which all numbers are replaced by the complex conjugate numbers. No physical result can depend upon which algebra is employed. If the operators of the dual algebra are written X^*, the correspondence between the two algebras is governed by the laws

$$(X + Y)^* = X^* + Y^*,\ (XY)^* = X^* Y^*,\ (\lambda X)^* = \lambda^* X^*.$$

The formation of the adjoint within the complex conjugate algebra is called transposition,

$$X^T = X^*\dagger = X\dagger^*.$$

It has the algebraic properties

$$(X + Y)^T = X^T + Y^T,\ (XY)^T = Y^T X^T,\ (\lambda X)^T = \lambda X^T.$$

The measurement symbols of a given description provide a basis for the representation of an arbitrary operator by N^2 numbers, and the abstract properties of operators are realized by the combinatorial laws of these arrays of numbers, which are those of matrices. Thus

$$X = \sum_{a'a''} \langle a'|X|a''\rangle M(a',a'')$$

defines the matrix of X in the a-description or a-representation, and the product

$$\begin{aligned} XY &= \Sigma\langle a'|X|a''\rangle M(a',a'')\Sigma\langle a^{\text{iv}}|Y|a'''\rangle M(a^{\text{iv}},a''') \\ &= \Sigma\langle a'|X|a''\rangle \delta(a'',a^{\text{iv}})\langle a^{\text{iv}}|Y|a'''\rangle M(a',a''') \end{aligned}$$

shows that

$$\langle a'|XY|a'''\rangle = \sum_{a''} \langle a'|X|a''\rangle\langle a''|Y|a'''\rangle.$$

The elements of the matrix that represents X can be expressed as

$$\langle a'|X|a''\rangle = trXM(a'',a'),$$

and in particular

$$\langle a'|X|a'\rangle = trXM(a').$$

The sum of the diagonal elements of the matrix is the trace of the operator. The corresponding basis in the dual algebra is $M(a',a'')^*$, and the matrices that represent X^* and X^T are the complex conjugate and transpose, respectively, of the matrix representing X. The operator $X\dagger = X^{T*}$, an element of the same algebra as X, is represented by the transposed, complex conjugate, or adjoint matrix.

The matrix of X is the mixed ab-representation is defined by

$$X = \sum_{a'b'} \langle a'|X|b'\rangle M(a',b')$$

where

$$\langle a'|X|b'\rangle = trXM(b',a').$$

The rule of multiplication for matrices in mixed representations is

$$\langle a'|XY|c'\rangle = \sum_{b'}\langle a'|X|b'\rangle\langle b'|Y|c'\rangle.$$

On placing $X = Y = 1$ we encounter the composition property of transformation functions, since

$$\begin{aligned}\langle a'|1|b'\rangle &= trM(b',a')\\ &= \langle a'|b'\rangle.\end{aligned}$$

If we set X or Y equal to 1, we obtain examples of the connection between the matrices of a given operator in various representations. The general result can be derived from the linear relations among measurement symbols. Thus,

$$\begin{aligned}\langle a'|X|d'\rangle &= trXM(d',a')\\ &= trX\sum_{b'c'}\langle c'|d'\rangle\langle a'|b'\rangle M(c',b')\\ &= \sum_{b'c'}\langle a'|b'\rangle\langle b'|X|c'\rangle\langle c'|d'\rangle.\end{aligned}$$

The adjoint of an operator X, displayed in the mixed ab-basis, appears in the ba-basis with the matrix

$$\langle b'|X\dagger|a'\rangle = \langle a'|X|b'\rangle^*.$$

As an application of mixed representations, we present an operator equivalent of the fundamental properties of transformation functions:

$$\begin{aligned}\sum_{b'}\langle a'|b'\rangle\langle b'|c'\rangle &= \langle a'|c'\rangle\\ \langle a'|b'\rangle^* &= \langle b'|a'\rangle,\end{aligned}$$

which is achieved by a differential characterization of the transformation functions. If $\delta\langle a'|b'\rangle$ and $\delta\langle b'|c'\rangle$ are any conceivable infinitesimal alteration of the corresponding transformation functions, the implied variation of $\langle a'|c'\rangle$ is

$$\delta\langle a'|c'\rangle = \sum_{b'}[\delta\langle a'|b'\rangle\langle b'|c'\rangle + \langle a'|b'\rangle\delta\langle b'|c'\rangle], \tag{13}$$

and also

$$\delta\langle a'|b'\rangle^* = \delta\langle b'|a'\rangle.$$

One can regard the array of numbers $\delta\langle a'|b'\rangle$ as the matrix of an operator in the ab-representation. We therefore write

$$\delta\langle a'|b'\rangle = i\langle a'|\delta W_{ab}|b'\rangle,$$

which is the definition of an infinitesimal operator δW_{ab}. If infinitesimal operators δW_{bc} and δW_{ac} are defined similarly, the differential property (13) becomes the matrix equation

$$\langle a'|\delta W_{ac}|c'\rangle = \sum_{b'}[\langle a'|\delta W_{ab}|b'\rangle\langle b'|c'\rangle + \langle a'|b'\rangle\langle b'|\delta W_{bc}|c'\rangle],$$

from which we infer the operator equation

$$\delta W_{ac} = \delta W_{ab} + \delta W_{bc}. \tag{14}$$

Thus the multiplicative composition law of transformation functions is expressed by an additive composition law for the infinitesimal operators δW.

On identifying the a- and b-descriptions in (14), we learn that

$$\delta W_{aa} = 0$$

or

$$\delta\langle a' | a'' \rangle = 0,$$

which expresses the fixed numerical values of the transformation function

$$\langle a' | a'' \rangle = \delta(a', a'').$$

Indeed, the latter is not an independent condition on transformation functions but is implied by the composition property and the requirement that transformation functions, as matrices, be nonsingular. If we identify the a- and c-descriptions we are informed that

$$\delta W_{ba} = -\delta W_{ab}.$$

Now

$$\begin{aligned} \delta\langle a' | b' \rangle^* &= -i\langle a' | \delta W_{ab} | b' \rangle^* \\ &= -i\langle b' | \delta W_{ab}\dagger | a' \rangle, \end{aligned}$$

which must equal

$$\delta\langle b' | a' \rangle = i\langle b' | \delta W_{ba} | a' \rangle,$$

and therefore

$$\delta W_{ab}\dagger = -\delta W_{ba} = \delta W_{ab}.$$

The complex conjugate property of transformation functions is thus expressed by the statement that the infinitesimal operators δW are Hermitian.

The expectation value of property A for systems in the state b' is the average of the possible values of A, weighted by the probabilities of occurrence that are characteristic of state b'. On using (9) to write the probability formula as

$$p(a', b') = trM(a')M(b'),$$

the expectation value becomes

$$\begin{aligned} \langle A \rangle_{b'} &= \sum_{a'} a' p(a', b') = trAM(b') \\ &= \langle b' | A | b' \rangle, \end{aligned}$$

where the operator A is

$$A = \sum_{a'} a' M(a').$$

The correspondence thus obtained between operators and physical quantities is such that a function $f(A)$ of the property A is assigned the operator $f(A)$, and the operators associated with a complete set of compatible physical quantities form a complete set of commuting Hermitian operators. In particular, the function of A that exhibits the value unity in the state a', and zero otherwise, is characterized by the operator $M(a')$.

The physical operation symbolized by $M(a')$ involves the functioning of an apparatus capable of separating an ensemble into subensembles that are distinguished by the various values of a', together with the act of selecting one subensemble and rejecting the others. The measurement process prior to the stage of selection, which we call a nonselective measurement, will now be considered for the purpose of finding its symbolic counterpart. It is useful to recognize a general quantitative interpretation attached to the measurement symbols. Let a system in the state c' be subjected to the selective $M(b')$ measurement and then to an A-measurement. The probability that the system will exhibit the value b' and then a', for the respective properties, is given by

$$\begin{aligned} p(a',b',c') &= p(a',b')p(b',c') = |\langle a'|b'\rangle\langle b'|c'\rangle|^2 \\ &= |\langle a'|M(b')|c'\rangle|^2. \end{aligned}$$

If, in contrast, the intermediate B-measurement accepts all systems without discrimination, which is equivalent to performing no B-measurement, the relevant probability is

$$\begin{aligned} p(a',1,c') &= |\langle a'|c'\rangle|^2 \\ &= |\langle a'|\sum_{b'} M(b')|c'\rangle|^2. \end{aligned}$$

There are examples of the relation between the symbol of any selective measurement and a corresponding probability,

$$p(a', \ ,c') = |\langle a'|M|c'\rangle|^2.$$

Now let the intervening measurement be nonselective, which is to say that the apparatus functions but no selection of systems is performed. Accordingly,

$$\begin{aligned} p(a',b,c') &= \sum_{b'} p(a',b')p(b',c') \\ &= \sum_{b'} |\langle a'|M(b')|c'\rangle|^2 \end{aligned}$$

which differs from

$$p(a',1,c') = |\sum_{b'}\langle a'|M(b')|c'\rangle|^2$$

by the absence of interference terms between different b' states. This indicates that the symbol to be associated with the nonselective B-measurement is

$$M_b = \sum_{b'} e^{i\varphi_{b'}} M(b')$$

where the real phases $\varphi_{b'}$ are independent, randomly distributed quantities. The uncontrollable nature of the disturbance produced by a measurement thus finds its mathematical expression in these random phase factors. Since a nonselective measurement does not discard systems we must have

$$\sum_{a'} p(a',b,c') = 1$$

which corresponds to the unitary property of the M_b operators,

$$M_b \dagger M_b = M_b M_b \dagger = 1.$$

It should also be noted that, within this probability context, the symbols of the elementary selective measurements are derived from the nonselective symbol by replacing all but one of the phases by positive infinite imaginary numbers, which is an absorptive description of the process of rejecting subensembles.

The general probability statement for successive measurements is

$$p(a',b', \ldots s',t') = |\langle a'|M(b') \ldots M(s')|t'\rangle|^2$$

which is applicable to any type of observation by inserting the appropriate measurement symbol. Other versions are

$$p(a', \ldots t') = \langle t'|(M(a') \ldots M(s'))\dagger(M(a') \ldots M(s'))|t'\rangle$$

and

$$p(a', \ldots t') = tr(M(a') \ldots M(t'))\dagger(M(a') \ldots M(t')),$$

each of which can also be extended to all types of selective measurements, and to nonselective measurements (the adjoint form is essential here). The expectation value construction shows that a quantity which equals unity if the properties A, $B, \ldots S$ successively exhibit, in the sinistral sense, the values a', b', $\ldots s'$, and is zero otherwise, is represented by the Hermitian[3] operator $(M(a') \ldots M(s'))\dagger$-$(M(a') \ldots M(s'))$.

Measurement is a dynamical process, and yet the only time concept that has been used is the primitive relationship of order. A detailed formulation of quantum dynamics must satisfy the consistency requirement that its description of the interactions that constitute measurement reproduces the symbolic characterizations that have emerged at this elementary stage. Such considerations make explicit reference to the fact that all measurement of atomic phenomena ultimately involves the amplification of microscopic effects to the level of macroscopic observation.

Further analysis of the measurement algebra leads to a geometry associated with the states of systems.

[1] This development has been presented in numerous lecture series since 1951, but is heretofore unpublished.

[2] Here we bypass the question of the utility of the real number field. According to a comment in THESE PROCEEDINGS, **44**, 223 (1958), the appearance of complex numbers, or their real equivalents, may be an aspect of the fundamental matter-antimatter duality, which can hardly be discussed at this stage.

[3] Compare P. A. M. Dirac, *Rev. Mod. Phys*, **17**, 195 (1945), where non-Hermitian operators and complex "probabilities" are introduced.

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INGRAM OLKIN