Foundations of Physics manuscript No.
(will be inserted by the editor)

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Quarks, Leptons, and Hopf Algebra Propagators

Received: 2/21/2011 / Accepted: date

Abstract The weak quantum numbers of the elementary fermions arise as particular representations of the Lie symmetry $SU(3) \times SU(2) \times U(1)$. Hopf algebras provide a generalization of Lie algebras with the advantage that they also naturally model the algebra of Feynman diagrams. The simplest Hopf algebras are the group algebras generated by finite groups such as the permutation group of three elements, P_3. The simplest Feynman diagrams are those that define propagators. In this paper we examine the propagators of the Hopf algebra generated by the permutation group of three elements, $C[P_3]$. The algebra consists of a 6-dimensional complex vector space, with basis given by the six elements of P_3. Multiplication is defined by the group multiplication. We show that the propagators of this algebra naturally contain the quarks and leptons with their weak hypercharge, weak isospin, and baryon quantum numbers. We show that an extension of the algebra gives spin-1/2 and the generations.

Keywords weak hypercharge · weak isospin · baryon number · Hopf algebra · generation

PACS 11.30.Hv

There are several motivations for using Hopf algebras to model elementary particles. They arise naturally in quantum mechanics. [1] Feynman graphs are organized with an algebra similar to the Hopf algebra on rooted trees. The partition function of quantum statistics gives rise to a Hopf algebra structure. And a type of Hopf algebra, quantum groups, are used in quantum physics to model the nonideal (multi-frequency) behavior of lasers. [2] They are used in analyzing broken symmetries in string theory. [3] And there are certain advantages in replacing a Lie symmetry with a Hopf algebra. [4]

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The Feynman graphs that characterize a particle are the propagators. The propagators generate a trivial Hopf algebra in that it remains linear: connecting a propagator to a propagator results in a propagator. That is, propagators satisfy the idempotency law $xx = x$. Applied to a group algebra, idempotency defines a set of coupled quadratic equations. The solution set can be thought of as containing all the possible results from renormalizing propagators in the algebra.

The process of dressing a bare propagator to physical form amounts to multiplying that propagator to a high power (while maintaining its magnitude). For a typical initial condition, this process leads to an attractor which must be an idempotent of the Hopf algebra. The process is fast on the computer and it allows us to distinguish between the idempotents of the Hopf algebra. We use these techniques to show that the quarks and leptons appear naturally in the Hopf algebra generated by the permutations on three elements.

In the first section we introduce the concept by analyzing the simplest nontrivial group/Hopf algebra $C[P_2]$, and show that its propagators have the quantum numbers of the leptons. The second section compares Hopf algebra propagators with pure density matrices and demonstrates how to write operators. Section three analyzes the group/Hopf algebra $C[P_3]$ and shows that its quantum numbers define the quarks and leptons. Section four discusses a Hopf algebra extension of $C[P_3]$ which models the generations. The conclusion discusses how these ideas fit into a general theory of the elementary particles.

1 Introduction: The Leptons

An easy example of a Hopf algebra is the complex group algebra of a finite group. Let $G = \{g_j\}$ be a group with n elements. Define the group product as $g_j g_k \in G$, and let e be the multiplicative identity so that $eg = ge$. The Hopf algebra $C[G]$ is defined as the n-dimensional complex vector space with the group elements $\{g_j\}$ as the basis. That is, given n complex numbers α_j, a general element of $C[G]$ is given by:

$$\phi_\alpha = \Sigma_j \alpha_j g_j. \tag{1}$$

Addition and scalar multiplication are defined as usual for a complex vector space. Multiplication of two Hopf algebra elements ϕ_α and ϕ_β is defined by using the group algebra product:

$$\phi_\alpha\ \phi_\beta = \Sigma_j \Sigma_k \alpha_j \beta_k (g_j g_k). \tag{2}$$

The multiplicative identity of the Hopf algebra is $1e = e$. This Hopf algebra models the controlled-NOT gate of qubit quantum information theory. For a more complete description see reference [5].

The smallest nontrivial group is the permutation group on 2 elements, $P_2 = \{e, s\}$, with $e^2 = s^2 = e$ and $es = se = s$. The regular representation of this group is:

$$\hat{e} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{s} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

The Hopf algebra elements are of the form:

$$(\alpha_e, \alpha_s) = \alpha_e e + \alpha_s s, \tag{4}$$

and the product of the elements (α_e, α_s) and (β_e, β_s) is

$$(\alpha_e, \alpha_s)(\beta_e, \beta_s) = (\alpha_e\beta_e + \alpha_s\beta_s, \alpha_e\beta_s + \alpha_s\beta_e). \tag{5}$$

If we arrange the elements in 2×2 matrices by $(\alpha_e, \alpha_s) \to \alpha_e\hat{e} + \alpha_s\hat{s}$, we can reproduce the above multiplication:

$$\begin{pmatrix} \alpha_e & \alpha_s \\ \alpha_s & \alpha_e \end{pmatrix} \begin{pmatrix} \beta_e & \beta_s \\ \beta_s & \beta_e \end{pmatrix} = \begin{pmatrix} \alpha_e\beta_e + \alpha_s\beta_s & \alpha_e\beta_s + \alpha_s\beta_e \\ \alpha_e\beta_s + \alpha_s\beta_e & \alpha_e\beta_e + \alpha_s\beta_s \end{pmatrix}. \tag{6}$$

The technique shown above generalizes; the Hopf algebra of a finite group G is obtained by treating the regular representation of G as the basis elements of a vector space, with addition and multiplication defined by matrix addition and multiplication.

In quantum field theory, we typically wish to compute the probability of an interaction defined by initial and final conditions. These conditions become the external lines of the diagrams. For each diagram compatible with those external lines, we compute a complex number. We add these complex numbers; then the probability is the squared magnitude of the sum.

Suppose that we have two types of Feynman diagrams, E_T and S_T, that correspond to the advance of time by $t = T$. The E_T diagrams cause no change in the system, while the S_T diagrams change some property of the system so that two consecutive S_T diagrams will cancel. We have two complex numbers, α_{eT} and α_{sT} that we obtain by summing over all of the E_T and S_T Feynman diagrams, respectively.

We would like to find the Feynman diagrams for the passage of time by $t = 2T$. There are two types of Feynman diagrams for $t = 2T$; the E_{2T} that correspond to no change to the system, and the S_{2T} that change the property. But probabilities only depend on squared magnitudes of sums of complex numbers so what we're really concerned with is only the complex numbers associated with these two types of Feynman diagrams, α_{e2T} and α_{s2T}.

In computing α_{e2T} and α_{s2T}, we are to consider all combinations of Feynman diagrams that are consistent with the beginning and ending states. Two consecutive E_T will leave the system unchanged so their product will contribute to E_{2T}. Also contributing will be two consecutive S_T. The contributions to S_{2T} will be an E_T and an S_T in either order. The result for the complex numbers will be:

$$\begin{aligned} \alpha_{e2T} &= \alpha_{eT}\alpha_{eT} + \alpha_{sT}\alpha_{sT}, \\ \alpha_{s2T} &= \alpha_{eT}\alpha_{sT} + \alpha_{sT}\alpha_{eT}. \end{aligned} \tag{7}$$

This is equivalent to the Hopf algebra multiplication of Eq. (6).

Suppose that T is a very short time and so our experiments are only concerned with the limit of α_{enT} and α_{snT} as $n \to \infty$. Since elementary particles are states that are stable (over times much shorter than their lifetime), we look for situations where α_{enT} and α_{snT} are stable. In the language of the Hopf multiplication, we require that

$$(\alpha_{enT}, \alpha_{snT})(\alpha_{enT}, \alpha_{snT}) = (\alpha_{enT}, \alpha_{snT}). \tag{8}$$

Multiplying a quantum state by an arbitrary complex phase leaves it unchanged so we will generalize the above to:

$$(\alpha_{enT}, \alpha_{snT})(\alpha_{enT}, \alpha_{snT}) = e^{i\theta}(\alpha_{enT}, \alpha_{snT}). \tag{9}$$

These amount to two quadratic equations in two unknowns. The four solutions of Eq. (8) are:

$$\begin{array}{l|l} & (\ \alpha_{enT}\ ,\ \alpha_{snT}\ ) \\ \hline & e^{i\theta}\ (\ 0\ ,\ 0\ ), \\ & e^{i\theta}\ (\ 1/2\ ,\ -1/2\ ), \\ & e^{i\theta}\ (\ 1/2\ ,\ +1/2\ ), \\ & e^{i\theta}\ (\ 1\ ,\ 0\ ). \end{array} \tag{10}$$

The weak hypercharge and weak isospin quantum numbers of the leptons are:

$$\begin{array}{c|cc} & t_0 & t_3 \\ \hline \bar{\nu}_L & 0 & 0 \\ \bar{\nu}_R & +1 & -1/2 \\ \bar{e}_R & +1 & +1/2 \\ \bar{e}_L & +2 & 0 \\ \hline \nu_L & 0 & 0 \\ \nu_R & -1 & +1/2 \\ e_R & -1 & -1/2 \\ e_L & -2 & 0 \end{array}. \tag{11}$$

These are equivalent to Eq. (10) by choosing:

$$\begin{aligned} \alpha_{enT} &= t_0/2, \\ \alpha_{snT} &= t_3, \\ \theta &= \pi \text{ for the leptons,} \\ & \quad 0 \text{ for anti-leptons.} \end{aligned} \tag{12}$$

The above correspondence amounts to treating the elementary particles as the long time propagators. The correspondence is natural in that it relates the $U(1)$ symmetry of weak hypercharge t_0, with the identity element of the Hopf algebra e, while the $SU(2)$ symmetry of weak isospin t_3, is related to the swap element s which is the Pauli spin matrix σ_x.

2 Hopf Algebra Quantum States

Let $|a\rangle$ be a quantum state represented by a state vector or spinor. An alternative representation is the pure density matrix:

$$\rho_a = |a\rangle\langle a|. \tag{13}$$

Such a matrix is idempotent:

$$\rho_a \rho_a = \rho_a. \tag{14}$$

The solutions to our the Hopf algebra long-time propagator equation Eq (8) are idempotent under the Hopf algebra multiplication. Pure density matrices are also Hermitian:

$$\rho_a^\dagger = \rho_a, \tag{15}$$

and have trace 1:

$$\mathrm{tr}(\rho_a) = 1. \tag{16}$$

These three requirements fully characterize the pure density matrices. That is, any matrix M which is idempotent, Hermitian, and has unit trace can be written as $M = |m\rangle\langle m|$.

Eliminating the Hermiticity requirement allows pure density matrices with differences between the bra and ket, i.e. in the form $|a\rangle\langle b|/|\langle a|b\rangle|$. Such states might be useful for modeling situations which do not have T symmetry. The trace of a Hopf algebra element is the number which multiplies the e component. The first and fourth solutions Eq. (10) have traces of zero and two, so we have to include states with trace other than unity. A matrix whose trace is not unity cannot be put into $|a\rangle\langle a|$ form; an example is the unit matrix. In the case of Hopf algebras this is not a problem as a state vector is not required.

We've interpreted e as the part of the quantum state that gives the quantum number for $t_0/2$ and s as that for t_3. In pure density matrices, the average value of an operator M for a state ρ is obtained by taking the trace:

$$\langle M\rangle = \mathrm{tr}\ (\rho M). \tag{17}$$

For the Hopf algebra, the trace is the e portion. For the value $\alpha_e e + \alpha_s s$, this is α_e:

$$\langle \alpha_e e + \alpha_s s\rangle = \mathrm{tr}\ (\alpha_e e + \alpha_s s) = \alpha_e. \tag{18}$$

Then the natural Hopf algebra choices for the weak hypercharge, weak isospin, and electric charge operators are:

$$\begin{aligned} T_0 &= 2e, \\ T_3 &= s, \\ Q &= T_0/2 + T_3 = e + s. \end{aligned} \tag{19}$$

These operators work as desired. The quantum numbers of the particle have been packed into the propagator.

3 Quarks

For the permutation group on 3 elements, we will use the elements $\{R, G, B\}$ and the group elements $\{e, j, k, r, g, b\}$ with their action defined as:

$$\begin{array}{c|ccc} & R & G & B \\ \hline e & R & G & B \\ j & G & B & R \\ k & B & R & G \\ r & R & B & G \\ g & B & G & R \\ b & G & R & B \end{array} \quad (20)$$

The permutation group is then

$$\begin{array}{c|cccccc} & e & j & k & r & g & b \\ \hline e & e & j & k & r & g & b \\ j & j & k & e & b & r & g \\ k & k & e & j & g & b & r \\ r & r & g & b & e & j & k \\ g & g & b & r & k & e & j \\ b & b & r & g & j & k & e \end{array} \quad (21)$$

The previous section showed that we need consider only the complex numbers. Accordingly, we will abuse the notation somewhat, eliminate α, n, and T from our notation, and use $\{e, j, k, r, g, b\}$ for the complex numbers. Where we need "e" for the base of the natural log, we will write it as an exponential.

The requirement that the long time propagators be stable is:

$$(e, j, k, r, g, b)(e, j, k, r, g, b) = e^{i\theta}(e, j, k, r, g, b), \quad (22)$$

where the multiplication is to be the Hopf multiplication defined by the permutation group on three elements. In order to keep the quantum numbers real (which amounts to requiring that the measurable quantities be real), we will restrict ourselves to the real values of $\exp(i\theta)$. That is, we will use $\theta = 0$ or π, as in the previous section. As before, this amounts to doubling the number of solutions by allowing negative quantum numbers.

In solving the equations, we can add the negative solutions back in at the end. Until then, Eq. (22) gives six quadratic equations in six unknowns:

$$\begin{aligned} e &= ee + jk + kj + rr + gg + bb, \\ j &= ej + je + kk + rg + gb + br, \\ k &= ek + jj + ke + rb + gr + bg, \\ r &= er + jg + kb + re + gk + bj, \\ g &= eg + jb + kr + rj + ge + bk, \\ b &= eb + jr + kg + rk + gj + be. \end{aligned} \quad (23)$$

The above reads directly from the group table Eq. (21). For example, there are six ways of obtaining e: $e = ee, jk, kj, rr, gg$, or bb. These become $e = ee+jk+kj+rr+gg+bb$ which can be abbreviated as $e = ee+2jk+r^2+g^2+b^2$.

As a first step in solving these equations, we rewrite them as an equivalent set of six equations:

$$\begin{aligned} e &= ee + 2jk + (r^2 + g^2 + b^2),\\ (r+g+b)^2 &= (e+j+k)(1-(e+j+k)),\\ 0 &= (j-k)(1+j+k-2e),\\ (1-3f+(e+j+k))r &= (r+g+b)(j+e),\\ (1-3f+(e+j+k))g &= (r+g+b)(j+e),\\ (1-3f+(e+j+k))b &= (r+g+b)(j+e). \end{aligned} \tag{24}$$

Choosing $e = 1/2$ and $j = -k$ solves the last four of these equations. The two remaining equations reduce to:

$$\begin{aligned} 1/2 &= \pm(r+g+b),\\ j^2 &= -1/8 + (r^2+g^2+b^2)/2. \end{aligned} \tag{25}$$

Since we have solved four equations with only two assignments, the solution space will be at least a 2-manifold. We will parameterize the solutions with complex numbers α, and β. We find four 2-manifolds of solutions:

e	j	k	r	g	b
$1/2$	$+\gamma$	$-\gamma$	$-1/6+\alpha$	$-1/6+\beta$	$-1/6-\alpha-\beta$
$1/2$	$-\gamma$	$+\gamma$	$+1/6-\alpha$	$+1/6+\beta$	$+1/6+\alpha+\beta$
$1/2$	$+\gamma$	$-\gamma$	$+1/6+\alpha$	$+1/6+\beta$	$+1/6-\alpha-\beta$
$1/2$	$-\gamma$	$+\gamma$	$-1/6-\alpha$	$-1/6+\beta$	$-1/6+\alpha+\beta$

, (26)

where $\gamma = \sqrt{\alpha^2+\beta^2+\alpha\beta-1/12}$.

Eliminating the case "$e = 1/2$ and $j = -k$", there are 12 discrete solutions. Six of these show up as two triplets:

e	j	k	r	g	b
$1/3$	$w^{+n}/3$	$w^{-n}/3$	0	0	0
$2/3$	$-w^{+n}/3$	$-w^{-n}/3$	0	0	0

(27)

where $w = \exp(2i\pi/3)$ and $n = 0, 1, 2$. The remaining six discrete solutions are:

e	j	k	r	g	b
0	0	0	0	0	0
1	0	0	0	0	0
$1/6$	$+1/6$	$+1/6$	$+1/6$	$+1/6$	$+1/6$
$5/6$	$-1/6$	$-1/6$	$-1/6$	$-1/6$	$-1/6$
$1/6$	$+1/6$	$+1/6$	$-1/6$	$-1/6$	$-1/6$
$5/6$	$-1/6$	$-1/6$	$+1/6$	$+1/6$	$+1/6$

(28)

All of our solutions appear in pairs that sum to $(e, j, k, r, g, b) = (1, 0, 0, 0, 0, 0)$. This is a general property of a group algebra and follows from the fact that $u^2 = u$ iff $(1-u)^2 = (1-u)$. Another general property is that each solution satisfies either $e+j+k+r+g+b = 0$ or $= 1$.

As before, we put $e = t_0/2$. For t_3, we previously used s, the element that satisfied $s^2 = e$. For P_3, there are three elements that square to e, so we put their sum as t_3. And with quarks, we have another quantum number, the baryon number, for which we will use the remaining elements, $j + k$:

$$\begin{aligned} e &= t_0/2, \\ r + g + b &= t_3, \\ e + r + g + b &= Q, \\ j + k &= B(q). \end{aligned} \tag{29}$$

The complete solution set for $\theta = 0$, with particle assignments, is:

$\theta = 0$	e	j	k	r	g	b
$\bar{\nu}_L$	0	0	0	0	0	0
d_L	1/6	1/6	1/6	$-1/6$	$-1/6$	$-1/6$
u_L	1/6	1/6	1/6	$+1/6$	$+1/6$	$+1/6$
$\bar{d}_L$	1/3	$w^{+n}/3$	$w^{-n}/3$	0	0	0
$\bar{\nu}_R$	1/2	$\pm\gamma$	$\mp\gamma$	$-1/6+\alpha$	$-1/6+\beta$	$-1/6-\alpha-\beta$
$\bar{e}_R$	1/2	$\pm\gamma$	$\mp\gamma$	$+1/6+\alpha$	$+1/6+\beta$	$+1/6-\alpha-\beta$
u_R	2/3	$-w^{+n}/3$	$-w^{-n}/3$	0	0	0
$\bar{Q}_{udL}$	5/6	$-1/6$	$-1/6$	$-1/6$	$-1/6$	$-1/6$
$\bar{Q}_{uuL}$	5/6	$-1/6$	$-1/6$	$+1/6$	$+1/6$	$+1/6$
$\bar{e}_L$	1	0	0	0	0	0

(30)

where n, w, α, β, and γ are as used in Eq. (27) and Eq. (26). The $\bar{Q}_{udL}$ and $\bar{Q}_{uuL}$ solutions have the quantum numbers of vector anti-diquarks (the third vector anti-diquark, $\bar{Q}_{ddL}$, has the same quantum numbers as the $\bar{u}_R$). Note that the baryon number assignment $B(q) = j + k$ only works for the $n = 1$ or 2, a choice we will ratify below.

With $\theta = \pi$, we obtain a similar table but with all numbers negated and with the particles and anti-particles swapped (and with L and R swapped):

$\theta = \pi$	e	j	k	r	g	b
ν_R	0	0	0	0	0	0
$\bar{d}_R$	$-1/6$	$-1/6$	$-1/6$	$+1/6$	$+1/6$	$+1/6$
$\bar{u}_R$	$-1/6$	$-1/6$	$-1/6$	$-1/6$	$-1/6$	$-1/6$
d_R	$-1/3$	$-w^{+n}/3$	$-w^{-n}/3$	0	0	0
ν_L	$-1/2$	$\mp\gamma$	$\pm\gamma$	$+1/6-\alpha$	$+1/6-\beta$	$+1/6+\alpha+\beta$
e_L	$-1/2$	$\mp\gamma$	$\pm\gamma$	$-1/6-\alpha$	$-1/6-\beta$	$-1/6+\alpha+\beta$
$\bar{u}_L$	$-2/3$	$+w^{+n}/3$	$+w^{-n}/3$	0	0	0
Q_{udR}	$-5/6$	$+1/6$	$+1/6$	$+1/6$	$+1/6$	$+1/6$
Q_{uuR}	$-5/6$	$+1/6$	$+1/6$	$-1/6$	$-1/6$	$-1/6$
e_R	-1	0	0	0	0	0

(31)

Thus the solutions to the long term propagator problem for this Hopf algebra correspond nicely to the weak quantum numbers of quarks and leptons. The remaining solutions correspond to various diquarks.

The mass (or Higgs) interaction would be particularly simple if we require that the differences between any pair of left and right handed particles carry the same charges (i.e. $H = e_L - e_R = d_L - d_R = \nu_R - \nu_L = u_R - u_L$):

$$\begin{aligned} H &= (3e \pm ij\sqrt{3} \mp ik\sqrt{3} - r - g - b)/6 \\ &= (1/2, \pm i\sqrt{3}/6, \mp i\sqrt{3}/6, -1/6, -1/6, -1/6). \end{aligned} \tag{32}$$

The above is the same quantum numbers as the $\bar{\nu}_R$. This defines the particle assignments up to the sign of the imaginary unit:

$\theta = 0$	t_0	$B(q)$	t_3
	$2e$	$j+k$	$r+g+b$
$\bar{\nu}_L$	0	0	0
d_L	1/3	1/3	−1/2
u_L	1/3	1/3	+1/2
$\bar{d}_L$	2/3	−1/3	0
$\bar{\nu}_R$	1	0	−1/2
$\bar{e}_R$	1	0	+1/2
u_R	4/3	1/3	0
$\bar{e}_L$	2	0	0

(33)

The $\theta = \pi$ assignments are similar.

From looking at the u_L and d_L, the W^+ must carry the quantum numbers:

$$W^+ = (r+g+b)/3 = (0, 0, 0, 1/3, 1/3, 1/3). \tag{34}$$

This is not included in the solutions Eq. (30) and so, in this theory, does not correspond to a (single) propagator. Indeed, in the standard model the W^+ is a massive particle and so must violate the $SU_c(3) \times SU_L(2) \times U_Y(1)$ symmetry.

4 Generations

Upon using the permutation group on three elements as a basis for a Hopf algebra, one naturally considers the Feynman diagrams that would give rise to that finite group. If the elementary fermions are made from three preons that can have three different conditions but can swap these positions, then the Feynman diagrams for their movement would give the permutation group on three elements. But no preon structure for the fermions has yet been detected. What has been observed is that the elementary fermions arise in three generations. In this section we will explore the possibility that the three generations arise from an extension of the Hopf algebra we've used to model the weak quantum numbers.

The first thing to note is that the preons suggested by our calculations have no internal structure. They are treated as scalar particles. But the spin-statistics theorem implies that we should use an odd spin particle for the preons. The simplest case is spin-1/2, but then we are faced with the problem of how to find three spin-1/2 states.

Quantum information theory (QIT) provides a possible solution to this problem. In QIT the concept of "complementary variables" is modeled in a finite dimensional Hilbert space by "mutually unbiased bases". [6] Two bases are mutually unbiased if the transition probabilities between states of the bases are all equal. For the example of spin-1/2, the probabilities must all be 1/2, so therefore two spin-1/2 bases $\{|a_1\rangle, |a_2\rangle\}$ and $\{|b_1\rangle, |b_2\rangle\}$ are mutually unbiased if:

$$|\langle a_j|b_k\rangle|^2 = 1/2, \quad \text{for all } j, k. \tag{35}$$

A "complete" set of mutually unbiased bases means a maximal set of bases. For an n-dimensional Hilbert space, the maximum number of MUBs is $n+1$. For spin-1/2, that means we can name at most three mutually unbiased bases. The usual choice is:

$$\{|+x\rangle, |-x\rangle\}, \;\; \{|+y\rangle, |-y\rangle\}, \;\; \{|+z\rangle, |-z\rangle\}, \tag{36}$$

that is, spin in the three perpendicular directions.

The transition probabilities between distinct elements of a single basis are zero, i.e. $|\langle +x|-x\rangle|^2 = 0$. So there are zero probabilities for particles making these transitions and we can exclude such pairs from our Hopf algebra basis states. This leaves just three basis states for spin-1/2:

$$\{|+x\rangle, |+y\rangle, |+z\rangle\}. \tag{37}$$

To get a Hopf algebra we need to allow a multiplication, that is, we need to upgrade our spinors to operators. We do this by going to the pure density matrix form. Putting the 3-dimensional vector $\mathbf{u} = (u_x, u_y, u_z)$ and using the Pauli spin matrices to represent spin-1/2, we have:

$$|u\rangle \to \rho_u = |u\rangle\langle u| = (1 + \mathbf{u}\cdot\boldsymbol{\sigma})/2 = \frac{1}{2}\begin{pmatrix} 1+u_z & u_x - iu_y \\ u_x + iu_y & 1-u_z \end{pmatrix}. \tag{38}$$

The three matrices $\{\rho_x, \rho_y, \rho_z\}$ do not generate a group because they are not closed under multiplication.

To close the group, consider all possible products of them. At first glance this appears to be an infinite set, but an operator is completely determined by its left and right-side projections. So the following set is "closed" under multiplication:

$$(\rho_j \rho_k), \;\; \text{for} \;\; j,k \in \{x,y,z\}, \tag{39}$$

if by "closed" we mean that any product is a complex multiple of an element of the set. Since our purpose for the group is to make a Hopf algebra, we can allow this generalization of closure. And such a generalization is natural for Feynman diagrams.

An arbitrary element of the algebra is given by:

$$\Sigma_{j,k}\alpha_{jk}\;(\rho_j\rho_k), \tag{40}$$

where α_{jk} are nine complex numbers. Multiplication uses the multiplication of Pauli algebra projection operators ρ_j. We call this the "Hopf algebra over the spin-1/2 MUBs". Note that since the algebra is defined using the pure density matrices or projection operators ρ_j, there are no arbitrary complex phases. However, complex Berry-Pancharatnam phases [7, 8, 9] are introduced.

As before, we solve for the elements of the algebra that satisfy $xx = x$ and therefore correspond to long time propagators (subject to the additional constraint that they have unit trace). The calculation is somewhat involved. [10] We find that the long time propagators appear as three copies of the usual spin-1/2 which we associate with the three generations. The form of the long time propagators suggests a justification for the lepton mass formulas of Yoshio Koide. [11, 12, 10]

The idempotent elements of our Hopf algebra $C[P_3]$ are the only possible results for the renormalization of propagators of the algebra. If one begins with a non-zero element of the algebra, and one repeatedly squares and normalizes it, the element quickly approaches one of the idempotents. If the initial value was taken with some particular symmetry, it is possible to obtain any one of the solutions. However, for a typical random element, one finds that one always obtains one of only four solutions: the d_L, the u_L, and the $\bar{d}_L$ or its complex conjugate. These are the primitive elements of the algebra. They annihilate each other (i.e. two distinct primitive idempotents multiply to give zero). And given any subset of them, the sum over that subset is an idempotent. Finally, they sum to unity. These are the algebraic properties that characterize a "complete set of primitive idempotents," the concept from pure density matrices that corresponds to a complete set of commuting observables.

For the Hopf algebra over the spin-1/2 MUBs, three idempotents (corresponding to the three generations) were found in [10]. These three are the primitive ones and are the only ones with a non zero volume in their domain of attraction. They form a complet set of primitive idempotents.

5 Conclusion

The elementary particle zoo includes many hundreds of spin-1/2 fermions. Which are modeled using the following propagator?

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\gamma_\mu p^\mu + 0.511 \text{ eV})}{p^2 - (0.511 \text{ eV })^2 + i\epsilon} e^{-ip\cdot(x-y)} \tag{41}$$

Of course the 0.511eV gives the game away. In general, the identity of an elementary particle is determined by its propagator. But in practice, for a previously unknown particle, one must specify both its propagator and its various quantum numbers. In this paper we've attempted to use Hopf algebras to bundle the charges into the same mathematical object as the propagators.

Traditionally, when we think of quantum states we imagine distinct states as being orthogonal. For example, the right handed and left handed electron e_L and e_R are assumed to be orthogonal. Orthogonality means that the transition probabilities are zero. Their propagators multiply to give zero. To get a non-zero probability we are forced to insert an interaction vertex between the propagators. By encoding the charges into the propagators, we can hope that there will be no information left for the vertices and the vertices can be set to unity: dyanmics from kinematics.

We're using two Hopf algebras. The first is generated by the permutation group of three elements. The long time propagators of this Hopf algebra have a structure that reminds one of the quantum numbers of the elementary quarks and leptons. The use of the permutation group suggests that the Feynman diagrams should be that of three inter-weaving strands, that is, three particles which swap three orientations amongst themselves. Since the permutation group knows nothing about spin this model is literal only for three scalar particles.

Our second Hopf algebra explores the behavior of a single strand of the first Hopf algebra. We use the mutually unbiased bases of the Pauli spin matrices to model the three orientations. We show that the long time propagators of this Hopf algebra come in three classes which naturally model the three elementary particle generations. Together, these two Hopf algebras describe the quarks and leptons, spin-1/2, and the generations they appear in.

To obtain a complete theory of elementary particles using Hopf algebra will require modeling the vertices. These seem likely to require Feynman diagrams with loops. Our hope is that a deeper understanding of how Hopf algebras model the renormalization of simple vertices will complete the model.

Acknowledgements The author would like to thank Marni Sheppeard for providing inspiration and assistance on this paper and in other, related, work.

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